Hjelmslev's Theorem

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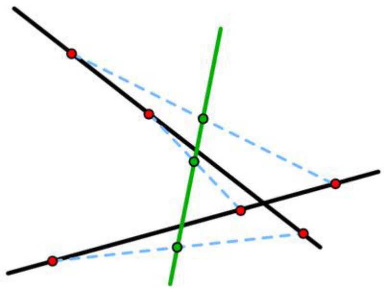

I came across this remarkable result in Futility Closet ([1]):

On each of these two black lines is a trio of red points marked by the same distances. The midpoints of segments drawn between corresponding points are collinear.

(Discovered by Danish mathematician Johannes Hjelmslev.)

This result seems amazing and mysterious. I wondered if I could think of a proof. I found a simple approach that did not use plane geometry. And suddenly, like a magic trick exposed, the result seemed obious.

Proof

This problem made me think of "The Four Travelers Problem",[1] which got me to thinking about the red points on the black lines as momentary positions of two moving points at the same speed. These two points would leave equal distances in each time interval, which satisfied the requirements of the theorem.

I parameterized the problem with vectors as shown in Figure 1. Imagine the two lowest red dots on the two black lines as the starting positions for moving dots given by the position vectors

$$\mathbf{P}_1(0) = (a, b) \text{ and } \mathbf{P}_2(0) = (c, d).$$

Then later positions for these points at some time t will be given by

$$\mathbf{P}_1(t) = (x_1, y_1) \text{ and } \mathbf{P}_2(t) = (x_2, y_2),$$

yielding equal distances

$$|\mathbf{P}_1(t) - \mathbf{P}_1(0)| = |\mathbf{P}_2(t) - \mathbf{P}_2(0)|.$$

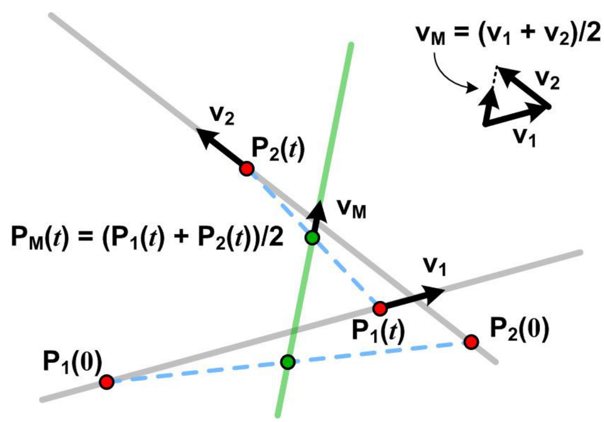

Figure 1

The velocity vectors for these moving points are the time derivatives of the position vectors or

$$\mathbf{v}_1 = \mathbf{P}_1'(t) = (\dot{x}_1, \dot{y}_1) \text{ and } \mathbf{v}_2 = \mathbf{P}_2'(t) = (\dot{x}_2, \dot{y}_2).$$

Now the midpoint on the line joining the two red points is given by the position vector $\mathbf{P}_M(t)$ that is the average of the position vectors of the two points, namely,

$$\mathbf{P}_M(t) = (\mathbf{P}_1(t) + \mathbf{P}_2(t))/2 = ((x_1 + x_2)/2, (y_1 + y_2)/2)$$

Therefore the velocity vector for the midpoint is

$$\mathbf{v}_M = \mathbf{P}_M'(t) = (\mathbf{P}_1'(t) + \mathbf{P}_2'(t))/2 = (\mathbf{v}_1 + \mathbf{v}_2)/2$$

[1] http://josmfs.net/2019/01/01/the-four-travelers-problem/

But since both points are moving at constant velocities, then so must the midpoint. That means its velocity vector is a constant (with constant direction) and so defines a straight line, which is what we wanted to prove.

Comment 1. Note that we did not really need the two points to move at the same speeds, only that they were moving at constant speed and direction, that is, constant velocities. Then the intervals traveled on the two black lines would be in a proportional relationship equal to the ratio of the speeds of the two moving points. So we could have a more general theorem that only requires the sets of intervals along the two black lines be in a proportional relationship, not necessarily equal to 1.

Comment 2. Wikipedia had an entry[2] regarding Hjelmslev's Theorem but no proof. I haven't bothered to look further.

References

[1] "Hjelmslev's Theorem," *Futility Closet*, 28 May 2024.
(https://www.futilitycloset.com/2024/05/28/hjelmslevs-theorem/, retrieved 6/2/2024)

[2] https://en.wikipedia.org/wiki/Hjelmslev%27s_theorem