Classic Geometry Paradox

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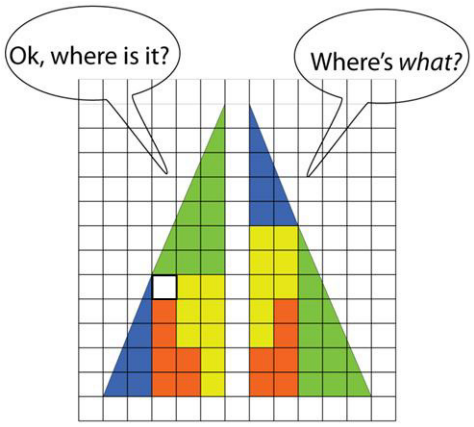

Coming across this classic geometric paradox recently in *Futility Closet* ([1]) motivated me to write down its solution in detail.

Where did the empty square come from?

In any case, this is the canonical example for why I avoid visual geometric proofs—you can be so easily fooled. Real proofs require plane or analytic geometry arguments.

Solution

The key to the solution is to realize the stacked colored geometric shapes do not exactly fill a triangle (Figure 1). The deviation (in red) is exaggerated in Figure 2 to show what is happening.

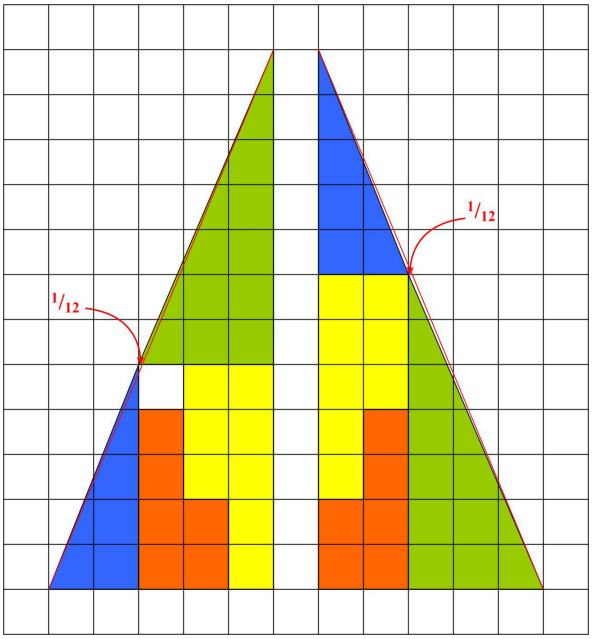

Figure 1

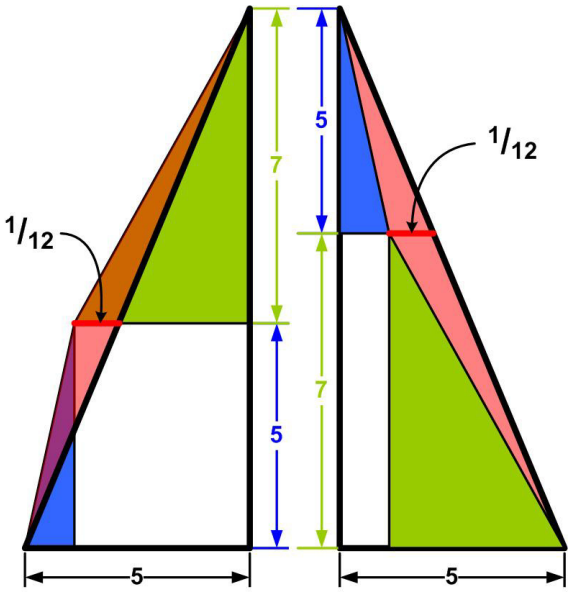

Figure 2

The supposed triangular boundary is shown in black in Figure 2. The triangle has an altitude of 12 units and base of 5 units. If the green triangle on the left hand side of Figure 2 were similar to the black triangle, then its base would be x, where

$$12/5 = 7/x \;\Rightarrow\; x = 35/12.$$

But that falls short of the actual base of 3 by Δx where

$$\Delta x = 3 - x = 36/12 - 35/12 = 1/12.$$

Similarly, if the blue triangle on the right hand side of Figure 2 were similar to the black triangle, then its base would be y, where

$$12/5 = 5/y \;\Rightarrow\; y = 25/12.$$

But that exceeds the actual base of 2 by Δy where

$$\Delta y = y - 2 = 25/12 - 24/12 = 1/12.$$

So the excess of the area of the shape on the left hand side, including the blank square, over the area of the black triangle is

$$½\ \Delta x\ 5 + ½\ \Delta x\ 7 = ½\ \Delta x\ 12 = ½.$$

And the deficiency of the area of the shape on the right hand side from that of the black triangle is

$$½\ \Delta y\ 7 + ½\ \Delta y\ 5 = ½\ \Delta y\ 12 = ½.$$

So the difference in area between the two shapes is

$$½ + ½ = 1,$$

the area of the phantom missing square. What a difference a tiny sliver makes.

References

[1] "Constitutional Crisis", *Futility Closet*, 17 April 2024 (https://www.futilitycloset.com/2024/04/17/constitutional-crisis/)