Learning Mathematics

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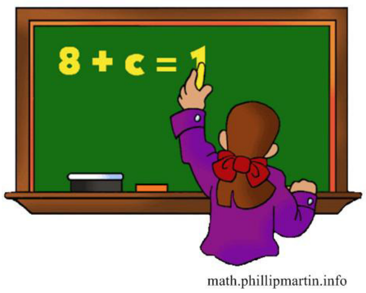

In one of our periodic FaceTime calls I found out that my granddaughter in 6th grade was interested in learning algebra and had gotten a book to help her out. Clearly this initiative to get a head start prior to the normal course curriculum excited me, so I wrote what I thought was an insightful essay on the meaning and purpose of algebra. Needless to say it was an abysmal failure.

That got me to thinking *deeply* about what it meant to learn mathematics and in particular symbolic algebra. I have alluded to a "mathematical mentality" off and on in various essays, but never really focused on what this meant for a fresh student starting out. This mathematical mindset reminded me of what a college professor said about Eastern thought's "enlightenment": your physical surroundings remain unchanged, but you see everything in an entirely different way. So it is difficult to explain this state to those who have not achieved it.

As I sought ways to bridge the gap, I looked for videos that would show how teachers are trying to help students make the transition. I came across a set sponsored by the Annenberg Foundation, which had supported the fantastic *Mechanical Universe* videos of the 1980s, and watched the first video on "Variables and Patterns of Change" ([1]). I was surprised to see the videos were already 20 years old (anything in the 21st century seems like yesterday to me), so I don't know if the methods are still being applied, but I assume so.

Concrete Example

Let me go through an example from this first video on algebra. It looks like the method was an extension of the Piaget[1] sticks idea that arose in the 1960s to provide a "hands-on" feel for the operations in arithmetic for elementary school children.

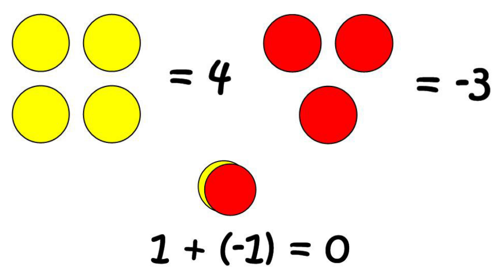

Figure 1

This time circular tokens are used for integers that were yellow on one side for positive numbers and red on the other for negative numbers. Putting a red token on top of a yellow token is equivalent to adding the negative of the token to the positive of the token yielding zero, that is, canceling the token (Figure 1).

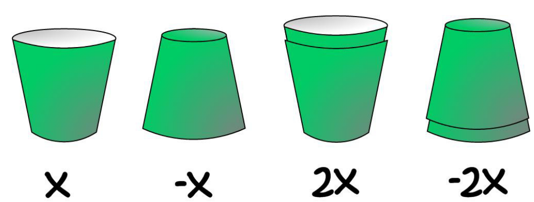

Figure 2

For the variable, or unknown, nested cups are used. A turned-up cup is positive and a turned-down cup is negative. Nested cups represent a multiple of the variable, either positive or negative, depending on the orientation of the cups (Figure 2).

The students are given an equation to solve. This amounts to getting a single cup all by itself using legal arithmetic operations, the same operation on both sides of the equation.

[1] https://en.wikipedia.org/wiki/Piaget%27s_theory_of_cognitive_development

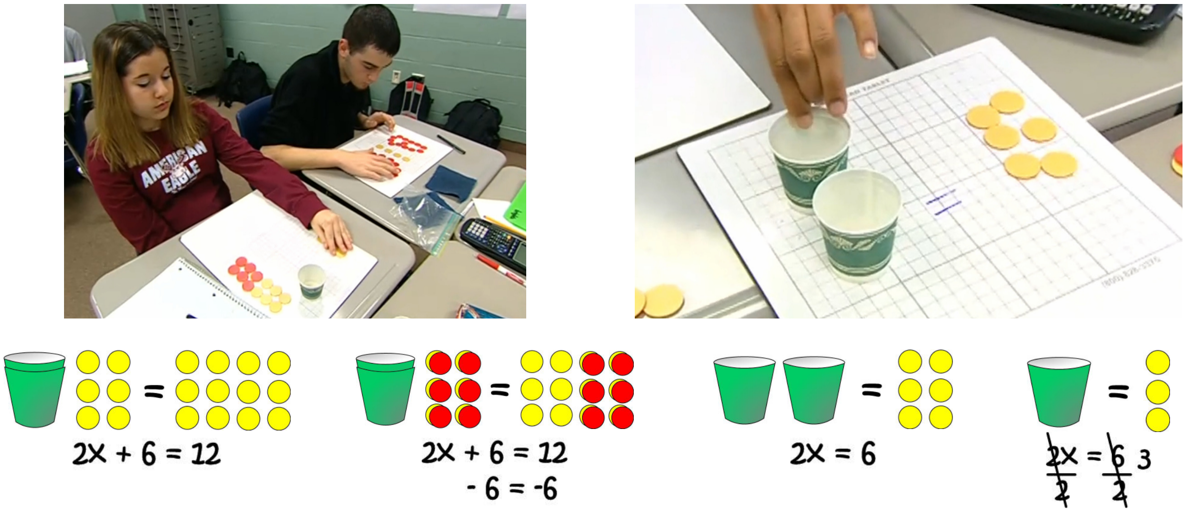

Figure 3

Figure 3 shows the example of solving for x in the equation $2x + 6 = 12$. Under each cups-and-tokens diagram is the corresponding algebraic equation and indicated operation, which the students are directed to write down as they go.

Commentary

My first reaction to this approach was that it was a tour de force—it was rather amazing to see how far one could go mimicking the algebraic manipulations with concrete objects, such as cups and tokens. Of course, it ran into difficulties when one of the examples ended with a fraction, which could not be represented by the tokens, unless they were broken.

Not being a psychologist I can't say whether this "concrete" approach is a help or a hindrance in learning algebra, at least initially. How helpful is it in forming mental pathways to solving math problems to learn these alternative "concrete" procedures? Some of the questions the teacher asked the students by way of characterizing what they saw in their current steps left me baffled, since I was not thinking "cups-and-tokens" but algebraically. I ultimately found the "cups-and-tokens" approach to be distracting. (I never bothered to learn how to use an abacus, since it required different mental procedures from performing the usual symbolic numerical manipulations.)

I admit I might be biased because I went through public school before all the "concrete" methods were used and just learned how to manipulate the numerical symbols. Yes, it was boring (so is practicing musical scales), but in those days we assumed the adults knew more than we did and that it was important to learn basic addition and multiplication tables for numerals (names of numbers), without having to constantly imagine we were dealing with apples or tokens or whatever. Yes, the numerical operations were introduced with concrete examples, but *we were not schooled into thinking the mathematical operations were always perfectly mirroring concrete behavior*. As the idea of a number evolved from a counting number to a negative number and then to a fraction, motivated by concrete examples, we learned to add, subtract, multiply, and divide them in their own way. Fractions, especially succumbed to the mechanical manipulation rules, since their physical analogs were not always evident.

This habit of "blind" manipulation of numbers (actually the numerals or symbols of numbers) followed the historical precedent leading up to the advent of symbolic algebra.[2] With the introduction of an unknown (a "thing" or *cosa* according to the Italians in the Renaissance) any type of number could be represented, even an irrational (non-integer, non-fraction). So all the mathematicians had to

[2] See my "Symbolic Algebra Timelines" (https://josmfs.net/symbolic-algebra-timelines/).

work with was arithmetic manipulations according to some rules. Remember the Greeks knew about irrational numbers (e.g. $\sqrt{2}$, the length of the hypotenuse of an isosceles right triangle of sides 1) and had handled them only with geometry. Including them in symbolic algebra manipulations was a big change.

Now visualization in mathematics is not to be disparaged (I primarily dwell in that realm). So it was another one of the great milestones in the evolution of mathematics that the connection between the basically abstract manipulations of numbers and symbols in symbolic algebra found a visual representation again in geometry, this time it was called "analytic geometry" in contrast to the venerable plane geometry or synthetic geometry of the Greeks. This connection is credited to Pierre de Fermat (1607-1665) and especially René Descartes (1596-1650). I don't want to digress here into a long discussion about the relationship between algebra and geometry, or the power that geometric visualization brings to understanding calculus. The point is that these endeavors lie in the realm of mathematics and do not *directly mirror* physical behavior. I have talked about this extensively in other essays, such as "Meditation on 'Is' in Mathematics Part II—Mathematical Reality".[3] True, there are amazing links between the *results* of mathematics and physical reality, but the mathematical manipulations involved follow mathematical principles and do not directly mirror physical transformations.

Critique

So is it misleading to begin a student's education in mathematics, especially advanced topics such as algebra, with constant references to physical models that are of limited scope? I sympathize with those who are visually oriented and find it difficult to retain the behavior of abstract structures and their manipulations without a mental picture. I always struggled with courses in abstract algebra that required memorizing the behavior of groups, rings, modules, algebras, fields, and so on. But these structures are powerful, organizing entities and reveal even more richness in the areas of math that are more visualizable, such as geometry, topology, manifolds, etc.

But at a more elementary level is such over-dependence on concreteness a problem? It seems to me the essence of algebra is inverse operations, not cups or tokens. That is, in order to dig out a variable from an equation where it has been buried by a bunch of operations, you want to apply their inverse operations to collect all the numbers on one side of the equation leaving the variable or unknown by itself on the other. This activity can be understood without any appeal to concrete examples of supposedly equivalent physical operations.

Keith Devlin certainly thinks there has been too much emphasis on the type of concrete thinking that works for counting numbers but not for the rest. He wrote a very provocative article in 2008 arguing that "multiplication is not repeated addition" ([2]), which early on stated (my emphasis):

> Let's start with the underlying fact. Multiplication simply is not repeated addition, and telling young pupils it is inevitably leads to problems when they **subsequently** learn that it is not. Multiplication of natural numbers certainly gives the same result as repeated addition, but that does not make it the same. Riding my bicycle gets me to my office in about the same time as taking my car, but the two processes are very different. Telling students **falsehoods** on the assumption that they can be **corrected later** is rarely a good idea. And telling them that multiplication is repeated addition definitely requires undoing later.
>
> How much later? As soon as the child progresses from whole-number multiplication to multiplication by fractions (or arbitrary real numbers). At that point, you have to tell a different story.
>
> "Oh, so multiplication of fractions is a DIFFERENT kind of multiplication, is it?" a bright kid

[3] p.6, https://josmfs.net/wordpress/wp-content/uploads/2019/01/Meditation-on-Is-in-Math-II-181224.pdf

> will say, wondering how many more times you are going to switch the rules. No wonder so many people end up thinking mathematics is just a bunch of arbitrary, illogical rules that cannot be figured out but simply have to be learned — only for them to have the rug pulled from under them when the rule they just learned is replaced by some other (seemingly) arbitrary, illogical rule.

Devlin eventually wrote seven more lengthy articles on the controversy he unleashed stretching to 2011. I have my own lengthy rebuttal to the extreme positions he takes, but that is for a another time. I will just say that I agree with his implication that too much of early math teaching is dependent on the tangible behavior of the counting numbers (a.k.a. natural numbers), which becomes a problem when other types of numbers are encountered. I think he goes too far when he says that "multiplication is repeated addition" is a *falsehood* and that it subsequently has to be *corrected*. He suggests in his papers that there is an ultimate, true idea of multiplication and implies that students should be taught that from the start. I disagree on two counts.

First, there is no ultimate definition of multiplication. As Devlin certainly knows, in abstract algebra the idea has morphed into countless directions and knows no end. One might argue that there is a final meaning of multiplication for *numbers*, but that requires knowing what we finally mean by a number, which really involves knowing what a *number system* is, and that turns out to be the abstract algebra construct called a *field*. But that certainly cannot be taught easily to school kids—or can it?

The second objection I have is with his claims of *falsehood* and *corrected* meaning. These assertions obscure the essential aspect of mathematics: it *evolves*. As the historical record shows, for centuries (millennia) human beings were more or less limited to the counting numbers, and so multiplication as repeated addition made perfectly good sense, and was *correct*. As those pesky negative whole numbers started appearing, the counting-number idea of multiplication got shaky: what did it mean to repeatedly add 5 to itself -3 times (or $\sqrt{2}$ times)? So the idea of multiplication had to evolve along with the idea of a number. This is what students should be taught: mathematics grows, and earlier notions need to be amended and considered in a new light, though the *old notions still apply to the old cases*. This is what makes mathematics so creative and powerful.

The student's criticism of math as consisting of arbitrary, illogical rules means there was a definite failure of understanding. The rules of arithmetic, as well as every other subject in math, are not at all arbitrary or illogical. There is a definite reason for them that is highly logical—and it does not rely on physical reality for its justification.

Conclusion

Unsurprisingly, it looks like I have not arrived a definitive answer to how to learn mathematics. I have a fondness for a roughly historically-based approach, since I think that parallels the evolving understanding of the individual. Just as mathematicians historically were able to justify including numbers beyond the counting numbers into a consistent structure, so can the modern individual. But that takes a maturing mathematical sophistication, especially the ability to be released from a dependency on physical reality for explanations. Mathematics is an abstract system unto itself. That it still retains valid correlations with physical reality is ultimately a fortunate mystery.

References

[1] "Variables and Patterns of Change", Part 1, *Insights Into Algebra 1: Teaching for Learning*, produced by Thirteen/WNET, Annenberg Lerner, Annenberg Foundation, 2004 (https://www.learner.org/series/insights-into-algebra-1-teaching-for-learning-2/) *Insights Into Algebra 1: Teaching for Learning* is an eight-part video, print, and web-based professional development workshop for middle and high school teachers. Participants will explore strategies to improve the way they teach 16 topics found in most Algebra 1 programs. In each session, participants will view two half-hour videos that showcase effective strategies for teaching mathematical topics. Then, led by the workshop guide, participants will engage in activities

designed to help them examine their teaching practice, incorporate what they are learning into their practice, share their experiences with other teachers, and reflect on their ongoing development.

[2] Devlin, Keith, “It Ain’t No Repeated Addition”, *Devin’s Angle*, Mathematical Association of America, June 2008. (https://www.maa.org/external_archive/devlin/devlin_06_08.html)