Additional Page Problem

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This is a clever puzzle from the 1986 AIME problems ([1]).

The pages of a book are numbered 1 through n. When the page numbers of the book were added, one of the page numbers was mistakenly added twice, resulting in an incorrect sum of 1986. What was the number of the page that was added twice?

Solution

Recall that the sum of consecutive whole numbers 1, 2, …, n is $n(n + 1)/2$. Since there is a repeated page, we should not expect an n such that $n(n + 1)/2 = 1986$. So we would like to start by finding the largest number of pages n such that $n(n + 1)/2 < 1986$. Then the difference should be a candidate for our missing page.

So first, let's find the number x such that $x(x + 1)/2 = 1986$ or $x^2 + x - 2\cdot1986 = 0$. Using the quadratic formula we have

$$x = \frac{-1 \pm \sqrt{1 + 8\cdot1986}}{2} = \frac{-1 \pm \sqrt{15889}}{2} = \frac{-1 + 126.05}{2}$$

where we have ignored a negative result. So we consider 126. But 126 – 1 is odd, and we need an even number to have n be an integer. So consider 125. Then $n = 124/2 = 62$. So the sum is $62\cdot63/2 = 1953$. This means the difference is $1986 - 1953 = 33$. So page 33 is a candidate for being counted twice in a sum of $n = 62$ pages.

We can't have a higher number than 126, but what about a lower number? The next odd number below 125 is 123, so that $n = 122/2 = 61$ and $61\cdot62/2 = 1891$. But $1986 - 1891 = 95$, and that would be page 95 out of $n = 61$ pages, which is impossible. So page 33 is our duplicate.

References

[1] "Problem 6" 1986 AIME Problems
(https://artofproblemsolving.com/wiki/index.php/1986_AIME_Problems)