Mystery Dice Question

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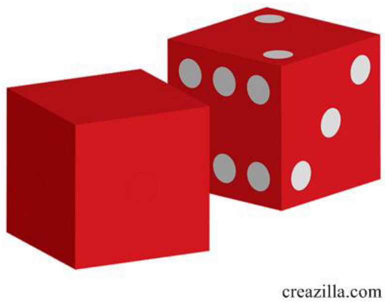
Jim Stevenson



This is a relatively simple probability question from Presh Talwalkar[1] that becomes an excuse to describe a powerful tool.

Amazon's Mystery Dice Interview Question

You are given a normal die and a blank die. (Each die is six-sided and equally likely to show each face). Label the blank die using the numbers 0 to 6 so that when you roll the two die the sum shows each whole number from 1 to 12 with equal chance. You can use a number more than once, or not at all, so you could label the faces 1, 2, 3, 4, 4, 5. But you do have to label all six faces of the blank die.

My Solution

You at least need a 6 on the blank die to be able to reach 12. If all the faces were 6, then all the values from 7 to 12 would be equally likely, but then there were be no instances of the values 1 to 6. We could get those if there were a 0 on a blank face. If half the blank faces were 0 and half were 6, then all the values from 1 to 12 would be equally likely. Specifically, since the dice are independent, we can multiply the probabilities of the outcomes on the separate dice to get the probability of each final outcome, namely, (1/2) (1/6) = 1/12. Again, showing each value from 1 to 12 is equally likely to result.

Talwalkar Solution

Talwalkar makes virtually the same argument for his initial solution. But then he uses the example as an opportunity to discuss the seemingly strange, but nifty, idea of probability-generating-functions. The references noted in the discussion fill in the details, but his discussion spells out the calculations for this example.

Probability generating functions.

I'm going to present the solution assuming basic familiarity with probability generating functions.[2] The probability generating function for a discrete random variable Z (over non-negative integers where $p_i = \Pr(Z = i)$ is the polynomial:

$$p_0x^0 + p_1x^1 + \ldots + p_kx^k$$

The key thing is we can find the sum of independent random variables—which is hard—by multiplying their probability generating functions—which is easier. So let's get started.

The probability generating function for a fair die is:

$$D(x) = (x + x^2 + x^3 + x^4 + x^5 + x^6)/6$$

The probability generating function for the uniform sum is:

$$U(x) = (x + x^2 + \ldots + x^{11} + x^{12})/12$$

[1] https://mindyourdecisions.com/blog/2019/07/11/how-to-solve-amazons-mystery-dice-interview-question/
[2] https://en.wikipedia.org/wiki/Probability-generating_function

We want to solve for the mystery die probability generating function:

$$M(x) = p_0x^0 + p_1x^1 + \ldots + p_6x^6$$

The probability generating function for the sum of two independent dice rolls will be the product of the generating functions.[3] Hence we have:

$$M(x)D(x) = U(x)$$

We can then divide by $D(x)$ to solve for $M(x)$:

$$M(x) = U(x)/D(x)$$

I can almost hear a student ask "But when will we ever need polynomial division?" Looks like it comes up in this problem, and that might help you get a job! It is straightforward to divide the two polynomials by long division, and the result is:

$$\begin{aligned} M(x) &= ((x + x^2 + \ldots + x^{11} + x^{12})/12)/((x + x^2 + x^3 + x^4 + x^5 + x^6)/6) \\ &= 0.5x^0 + 0.5x^6 \\ &= (3/6)(x^0) + (3/6)(x^6) \end{aligned}$$

We can now recover the probabilities from the generating function. We need 3 faces to be 0, and we need 3 faces to be 6.

So we get the same answer as before, but in a more systematic way. I think this is a pretty neat interview question!

As a fun note, the generating function answer is $0.5x^0 + 0.5x^6$. So actually we will get the sums 1 to 12 with equal chance for any die that has the same number n of faces of 0 and 6—whether it is {0, 6} or {0, 0, 6, 6}, etc.

Source

Careercup: https://www.careercup.com/question?id=3227685

[3] JOS: This is proved in https://www.cl.cam.ac.uk/teaching/0708/Probabilty/prob06.pdf. This reference gives a more expanded discussion of the generating function and at the end of the chapter discusses directly the probabilities for the sum of two dice using generating functions.