Circles in Circles

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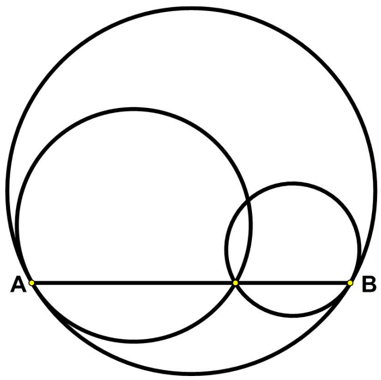

Here is another problem from the “Challenges” section of the *Quantum* magazine ([1]).

Inside a circle there are two intersecting circles. One of them touches the big circle in point A, the other in point B. Prove that if segment AB meets the smaller circles at one of their common points, then the sum of their radii equals the radius of the big circle. Is the converse true? (A. Vesyolov)

My Solution

I added the centers of the circles to the plot (Figure 1) as well as the radii. Since the two small circles are tangent to the large (blue) circle, their radii are perpendicular to the common tangent lines and so collinear with the radius of the large circle. Furthermore, since the point of intersection C lies on both small circles, it is joined to the respective centers by radii and thus forms two isosceles triangles as shown in the figure.

Since the centers of the two small circles lie on the radius of the large circle all the vertex angles are equal and so all the three triangles formed are similar. This means all the respective sides of the triangles are parallel and so the exposed blue region of the large triangle is a parallelogram. This shows immediately that the sum of the radii of the two smaller circles equals the large circle radius.

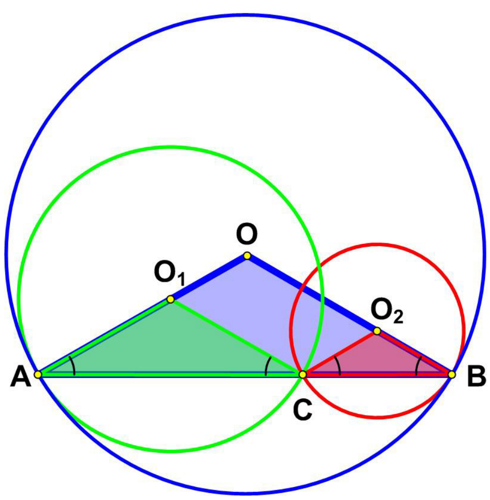

Figure 1 My Solution

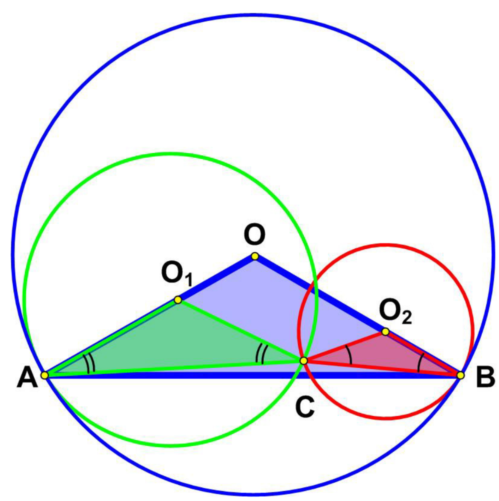

Figure 2 Converse

Converse. If we assume we have intersecting circles tangent to the large circle at points A and B, then we want to show the line AB passes through the intersection point C if the sum of the radii equals the large radius (Figure 2). Now if the large radius is the sum of the smaller radii (or equivalently, the blue quadrilateral has opposite sides equal), then the blue quadrilateral is a parallelogram. This is proved in Figure 3, since the corresponding triangles are congruent and so

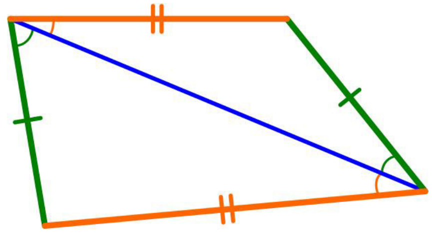

Figure 3 Sides Equal ⇒ Parallelogram

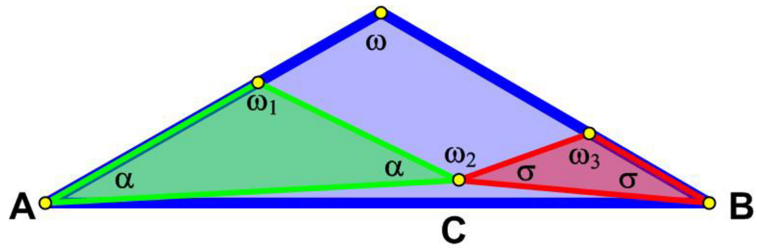

Figure 4 Parallelogram ⇒ ACB Straight Line

their corresponding angles equal, which implies the opposite sides are parallel. If the blue quadrilateral in Figure 4 is a parallelogram, then $\omega = \omega_1 = \omega_2 = \omega_3$. This implies all the triangles are similar ($\therefore\ \alpha = \sigma$ = angles at A and B of blue triangle) and $180^\circ = \alpha + \omega + \sigma = \alpha + \omega_2 + \sigma$, which means ACB is the straight line AB.

Quantum Solution

Denote by O, O_1, O_2 the centers of the given circles and by r, r_1, r_2 their respective radii (Figure 5). If segment AB meets circles O_1 and O_2 at their common point C, then isosceles triangles OAB, O_1AC, O_2BC are similar to one another (triangles OAB and O_1AC, say, have a common angle at vertex A). So the opposite sides of quadrilateral OO_1CO_2 are parallel and, therefore, congruent (by a property of the parallelogram). It follows that

$$r = OA = O\,O_1 + O_1A = O_2C + O_1A = r_1 + r_2.$$

The converse is also true: if $r = r_1 + r_2$, then segment AB passes through one of the common points of the smaller circles. To prove it, construct the parallelogram $OO_1'CO_2'$, whose vertices C' and O_2' lie on segments AB and OB, respectively (clearly this can always be done in a unique way). Triangles O_1AC' and $O_2'C'B$ are similar to isosceles triangle OAB, so

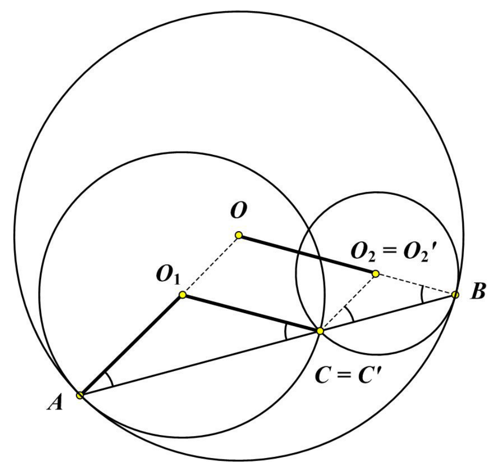

Figure 5 *Quantum* Solution

$$O_1C' = O_1A = r,$$

which means that C' lies on circle O_2, and

$$O_2'B = O_2'C' = OO_1 = OA - O_1A = r - r_1 = r_2$$

which means that $O_2' = O_2$ and that C' lies on circle O_2, which is what we had to prove. (V. Dubrovsky)

References

[1] “Challenges” M53 *Quantum Magazine*, Vol. 2 No. 5, National Science Teachers Assoc., Springer-Verlag, May-Jun 1992 p.19