# Elliptical Medians Problem

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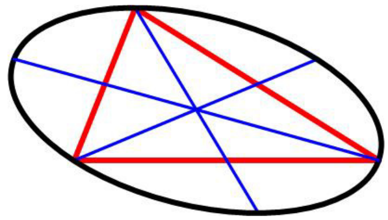

This is a tantalizing problem from the 1977 *Crux Mathematicorum* ([1]).

**278**. *Proposed by W.A. McWorter, Jr., The Ohio State University.*

If each of the medians of a triangle is extended beyond the sides of the triangle to 4/3 its length, show that the three new points formed and the vertices of the triangle all lie on an ellipse.

## My Solution

First, we need to consider some properties of medians of triangles, namely,

**Prop 1.** The three medians of a triangle meet at a common point inside the triangle.

**Prop 2.** The common point of intersection of the medians is 2/3 of their length from their respective vertices.

Since I didn't recall why these facts were true, I thought I would try to prove them here.

**Prop 1 Proof.** Consider two of the medians and their point of intersection in the triangle (Figure 1). Draw lines through the intersection point parallel to the sides of the triangle the medians intersect. Finally consider the two yellow triangles in Figure 1. By similar triangles the newly drawn lines are also bisected by the medians, and so the two triangles are congruent, since they have a common vertical angle.

Now draw a third line parallel to the last side of the triangle through the intersection point of the two medians, and construct the two new triangles formed as in Figure 2. These triangles are easily seen to be congruent to the previous two (equal angles and common side), and so the new horizontal line is bisected by the intersection point. This means the (blue) line from the third vertex through the intersection point also bisects the third side (via similar triangles and common scale factor) and so is a median.

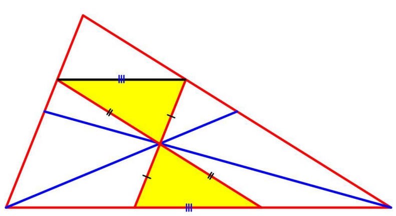

**Figure 1**

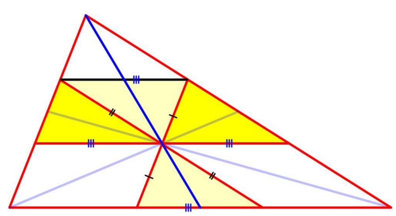

**Figure 2**

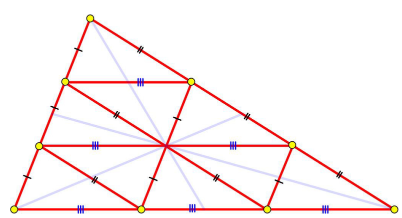

**Figure 3**

**Prop 2 Proof.** Now add the remaining parallel line segments to create 9 congruent triangles as shown in Figure 3. These parallel lines divide the sides of the triangle into 3 equal intervals, and so also divide the medians into 3 equal segments. Therefore the common point of intersection of each median is 2/3 of their length from their respective vertices.

Now back to the original problem.

Set the triangle so that its bottom edge is horizontal and the median intersection point is at the origin of a coordinate system (Figure 4). Then the coordinates of the three vertices A, B, C are given as $(x_1, y_1)$, $(x_2, y_2)$, $(x_3, y_3)$ relative to this origin. The general ellipse equation centered at the origin is

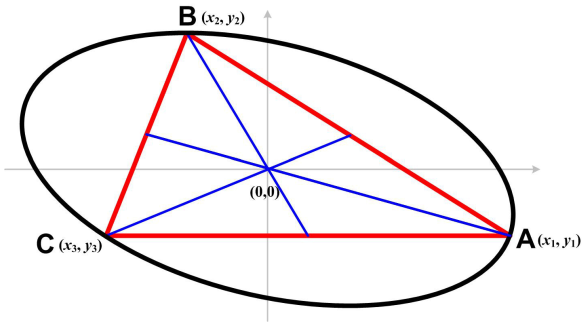


**Figure 4**

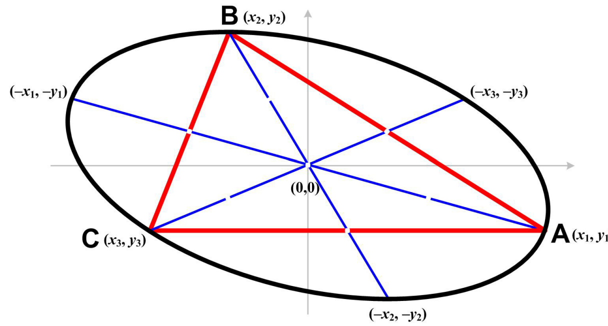


**Figure 5**

$$ax^2 + bxy + cy^2 = 1$$

after dividing through by the constant term. So the three points should give three linear equations in three unknowns *a, b, c*, which should determine the ellipse, uniquely.

A priori there is a question as to whether the center of the constructed ellipse is actually at the intersection point of the medians (as is shown in Figure 4). Notice if $(x, y)$ is on the ellipse, then so is its antipodal point $(-x, -y)$, that is, the point diametrically opposite $(x, y)$ through the origin. The center of the ellipse is also the midpoint of the line joining $(x, y)$ and $(-x, -y)$, which is given by

$$\left(\frac{x+(-x)}{2}, \frac{y+(-y)}{2}\right) = (0,0)$$

Therefore the intersection of the medians is the center of the ellipse (Figure 5).

The medians lie along the lines joining the vertices at A, B, and C to their antipodal points (Figure 5). Since the origin is the midpoint of these antipodal point lines, and the medians are each divided into 3 equal segments with the origin as one of the segment endpoints, the total length of the antipodal point lines is 4 times the lengths of the corresponding segments, that is, 4/3 the lengths of the corresponding medians—which is what we wanted to show.

## *Crux Mathematicorum* Solution

I thought some type of transformation from a circle might be an approach, but I couldn't remember what that might be. (No diagrams were included in the *Crux Mathematicorum* Solution.)

I. *Solution and comment by Dan Pedoe, University of Minnesota, Minneapolis.*

To remove all ambiguity in the wording of this problem, it should be understood that if the median AM, say, of triangle ABC is extended to A', then AA' = 4/3 AM.

An affine transformation will map triangle ABC onto an equilateral triangle, midpoints being mapped onto midpoints; and if G' is the centroid of the equilateral triangle, then a circle with centre G' evidently goes through the six designated points.[1] Hence an ellipse, in the original figure, goes through the vertices of triangle ABC and the tips of the extended medians.

This problem is discussed in my film *Central Similarities*, which I made for the Minnesota College Geometry Project. There is an intimate connection between the ellipse discussed above and the ellipse which touches the sides of triangle ABC at their midpoints (See Problem 318 [1978: 36]. (Editor)). One arises from the other by a central similarity (homothety) with centre at the centroid of the triangle and a suitable scale factor.

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[1] JOS: "evidently"? Why exactly? I think an argument similar to mine is needed relating the center to the length of the medians.

II. *Comment by O. Bottema, Delft, The Netherlands.*[2]

The factor 4/3 is the only one for which the theorem holds, and furthermore the ellipse concerned is Steiner's circumscribed ellipse, in which the tangents at the vertices are parallel to the opposite sides.

## References


[1] McWorter, Jr., W.A., "Problem 278," *Crux Mathematicorum*, Vol.3 No.8 Oct, Canadian Mathematical Society, 1977. p.227

[2] McWorter, Jr., W.A., "Problem 278 Solution," *Crux Mathematicorum*, Vol.4 No.4 Apr, Canadian Mathematical Society, 1978. p.109




[2] JOS: Well what do you know! This must be the mathematician of Bottema's Theorem fame.