Escalator Puzzle

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This is a problem from the 1987 American Invitational Mathematics Exam (AIME) ([1]).

Al walks down to the bottom of an escalator that is moving up and he counts 150 steps. His friend, Bob, walks up to the top of the escalator and counts 75 steps. If Al's speed of walking (in steps per unit time) is three times Bob's walking speed, how many steps are visible on the escalator at a given time? (Assume that this value is constant.)

My Solution

Figure 1 shows the problem setup where v_A is Al's speed in steps per second and v_B is Bob's speed. We are given that $v_A = 3v_B$. The corresponding times they are on the escalator are T_A and T_B respectively. The speed of the escalator is v_E. Let S be the (constant) number of steps visible at any time. Then we have the following equations that represent the relationships in the problem (which I hope are self-explanatory).

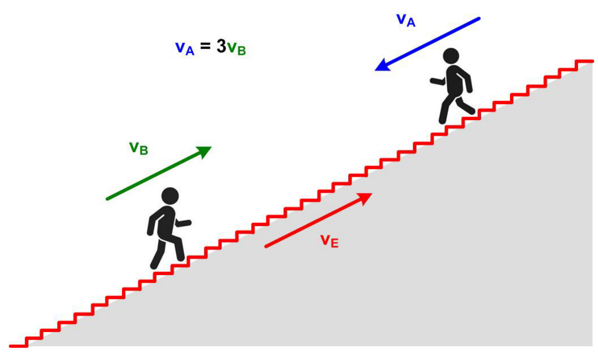

Figure 1

$$v_A T_A = 150 = S + v_E T_A$$

$$v_B T_B = 75 = S - v_E T_B$$

Then from the left hand sides of the two equations we get

$$v_A = 3v_B \quad \Rightarrow \quad T_A = (^2/_3)T_B.$$

Substituting this into the right hand side of the first equation, we get

$$150 = S + v_E(^2/_3)T_B \quad \Rightarrow \quad 3\cdot 75 = (^3/_2)S + v_E T_B$$

which, when added to the right hand side of the second equation (canceling the $v_E T_B$'s), yields

$$4\cdot 75 = (^5/_2)S$$

or

$$S = (^2/_5)\cdot 4\cdot 75 = 120 \text{ steps}$$

AIME Solutions

Solution 1

Let the total number of steps be *x*, the speed of the escalator be *e* and the speed of Bob be *b*.

In the time it took Bob to climb up the escalator he saw 75 steps and also climbed the entire escalator. Thus the contribution of the escalator must have been an additional *x* – 75 steps. Since Bob and the escalator were both moving at a constant speed over the time it took Bob to climb, the ratio of their distances covered is the same as the ratio of their speeds, so

$$\frac{e}{b} = \frac{x - 75}{75}.$$

Similarly, in the time it took Al to walk down the escalator he saw 150 steps, so the escalator must have moved 150 – x steps in that time. Thus

$$\frac{e}{3b} = \frac{150 - x}{150}$$

or

$$\frac{e}{b} = \frac{150 - x}{50}.$$

Equating the two values of e/b we have

$$\frac{x - 75}{75} = \frac{150 - x}{50}$$

and so $2x - 150 = 450 - 3x$ and $5x = 600$ and $x = 120$, the answer.

Solution 2

Again, let the total number of steps be x, the speed of the escalator be e and the speed of Bob be b (all "per unit time").

Then this can be interpreted as a classic chasing problem: Bob is "behind" by x steps, and since he moves at a pace of $b + e$ relative to the escalator, it will take

$$\frac{x}{3b + e} = \frac{75}{e}$$

time to get to the top. Similarly, Al will take

$$\frac{x}{3b - e} = \frac{150}{e}$$

time to get to the bottom.

From these two equations, we arrive at

$$150 = \frac{ex}{3b - e} = 2 \cdot 75 = \frac{2ex}{b + e} = \frac{6ex}{3b + 3e} = \frac{ex - 6ex}{(3b - e) - (3b + 3e)} = \frac{5x}{4}$$

$$\Rightarrow \quad 600 = 5x \quad \Rightarrow \quad x = 120,$$

where we have used the fact that

$$\frac{a}{b} = \frac{c}{d} = \frac{a \pm c}{b \pm d}$$

(the proportion manipulations are motivated by the desire to isolate x, prompting the isolation of the 150 on one side, and the fact that if we could cancel out the b's, then the e's in the numerator and denominator would cancel out, resulting in an equation with x by itself).

Solution 3

Let e and b be the speeds of the escalator and Bob, respectively. When Al was on his way down, he took 150 steps with a speed of $3b - e$ per step. When Bob was on his way up, he took 75 steps with a speed of $b + e$ per step.

Since Al and Bob were walking the same distance, we have $150(3b - e) = 75(b + e)$. Solving gets the ratio $e/b = 3/5$. Thus while Bob took 75 steps to go up, the escalator contributed an extra $(3/5)75 = 45$ steps.

Finally, there is a total of $75 + 45 = 120$ steps in the length of the escalator.

Solution 4

Please understand the machinery of an escalator before proceeding to read this solution.

Let the number of steps that disappear at the top of the escalator equal x. Assume that Al takes 3 steps per second and that Bob takes 1 step per second. Since Al counts 150 steps, it takes him 150/3 = 50 seconds to traverse the distance of the escalator moving downwards. Since Bob counts 75 steps, it takes him 75/1 = 75 seconds to traverse the distance of the escalator moving downwards.

For the sake of this solution, we activate the emergency stop button on the escalator.

Now, the escalator is not moving, or is simply a staircase. Imagine that Al is taking3 steps downwards every second, but we throw hands at him immediately after each second, such that he flinches and moves himself backwards x steps. This is equivalent to Al taking $3 - x$ steps downwards every second. Since we discovered that it takes him 50 seconds to get from the top to the bottom of the escalator, and we are forcing Al to imitate the movement of the escalator, it also takes him 50 seconds to move from the top to the bottom of the staircase. Thus, Al takes a total of

$$(3 - x)\cdot 50 = 150 - 50x \qquad (*)$$

steps.

The explanation for Bob is similar except now we pick him up and place him forward $x$steps immediately after he takes his usual step per second, and since we discovered he does this for 75 seconds, it takes him

$$(1 + x)\cdot 75 = 75 + 75x \qquad (**)$$

steps to get from the bottom to the top.

Note that because the escalator is broken and is now a staircase, Al and Bob must have had to take an equal amount of steps to get from the bottom to the top or from the top to the bottom. (Clearly, there are an equal amount of steps from the bottom to the top, and from the top to the bottom.) Therefore, we may equate equations (*) and (**) to get

$$150 - 50x = 75 + 75x \;\Rightarrow\; x = 3/5.$$

Therefore, substituting x in the expression we discovered in equation (*), Al takes a total of

$$150 - 50x = 150 - 50(3/5) = 120$$

steps, and we are done.

References

[1] "Problem 10" 1987 AIME Problems
(https://artofproblemsolving.com/wiki/index.php/1987_AIME_Problems)