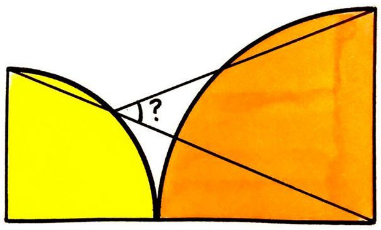
Curious Sunbeam Problem

27 April 2023

Jim Stevenson



This is a Catriona Agg problem[1] presented by itself, since it turned out to be the most challenging one I ever tried. Usually I can solve her problems in a few minutes or maybe hours, or sometimes days if they are especially challenging. But this problem has taken me weeks and I had to rely on a non-geometric argument. The problem is full of fascinating and unexpected relationships, but I couldn't find a way to use them to prove the answer.

Solution

First I will give my solution and then discuss a number of the relationships that show up. The first step will be to prove a surprising result that plays a fundamental role in my solution.

As shown in Figure 1, consider the (heavy black) line from the top of the quarter circle through an arbitrary point P on the quarter circle, along with the two isosceles triangles with base angles α and β. Then we *always* have $\alpha + \beta = 135°$, implying that the black line makes a 45° angle with respect to the β-triangle's base. This is true no matter where P falls.

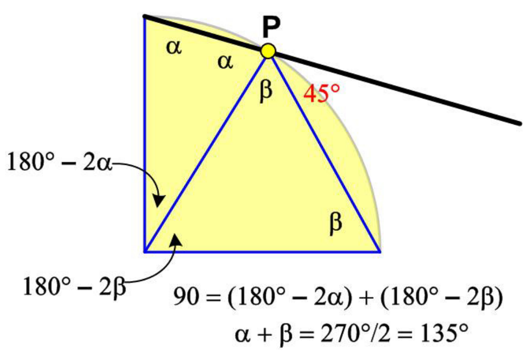


Figure 1

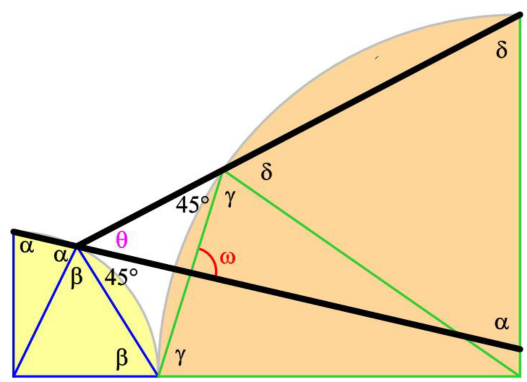


Figure 2

We use this general property to parameterize our problem as shown in Figure 2, where the large orange quarter circle is just the mirror image of the pattern shown in Figure 1. The figure also shows the intersection of the heavy black lines at an arbitrary point on the smaller quarter circle. Notice that the "α-line" does not necessarily pass through the center of the large quarter circle in this general case. The unknown angle we are solving for is designated θ and an intermediate angle is designated ω. Notice further that $\omega \geq \gamma$.

There are two extreme cases: when P is at the top of the smaller quarter circle ($\alpha = 90°$ and $\beta = 45°$) and when P is on the baseline ($\alpha = 45°$ and $\beta = 90°$). We approach both theses extremes in the limit to see what is happening.

The first case is shown in Figure 3. In that case the lower heavy black line is horizontal and thus parallel to the baseline. This means $\omega = \gamma < 90°$, since, as angles in a triangle, $2\gamma < 180°$.

[1] 16 Feb 2022 (https://twitter.com/Cshearer41/status/1493870019318632451)

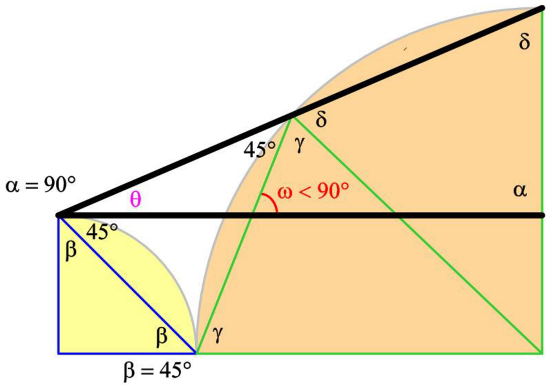


Figure 3

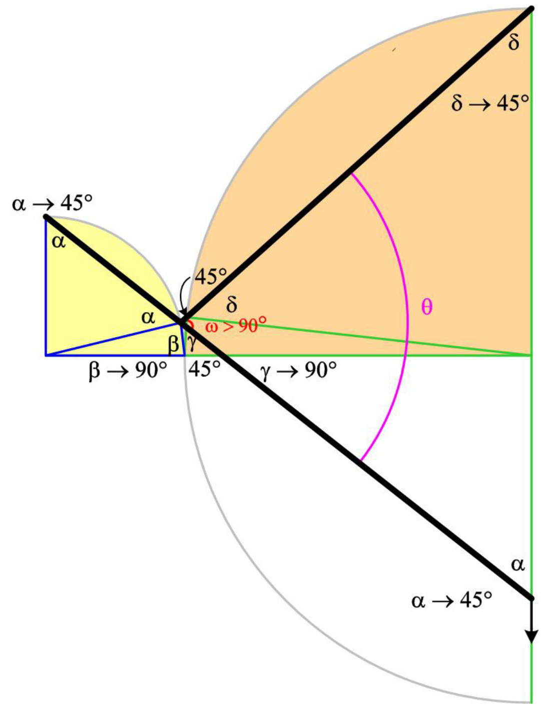


Figure 4

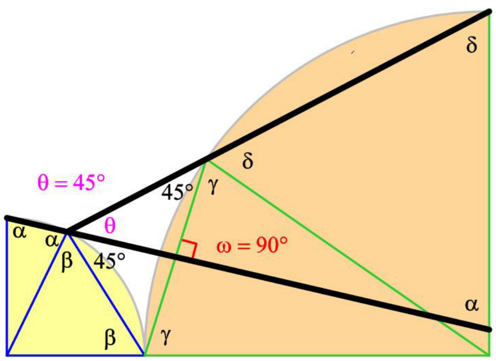


Figure 5

The second extreme case is more difficult to show and so is given as a limit in Figure 4. As $\gamma \rightarrow 90°$, $\gamma < \omega \rightarrow 135°$, which is greater than 90°.

Therefore by the Intermediate Value Theorem (IVT)[2] there must by a case where $\omega = 90°$. And since ω increases monotonically as the intersection point P moves along the small quarter circle, $\omega = 90°$ at only one place, and that also means $\theta = 45°$ there (Figure 5). But we don't know yet if the α-slant line goes through the center of the large quarter circle for this value of ω.

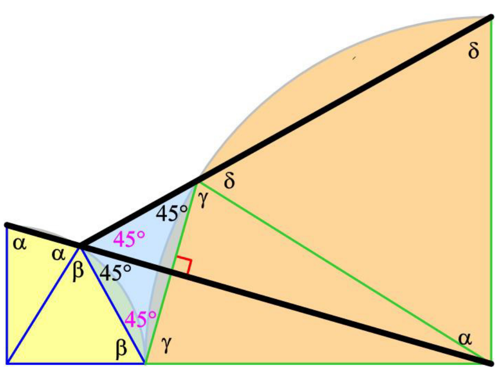


Figure 6

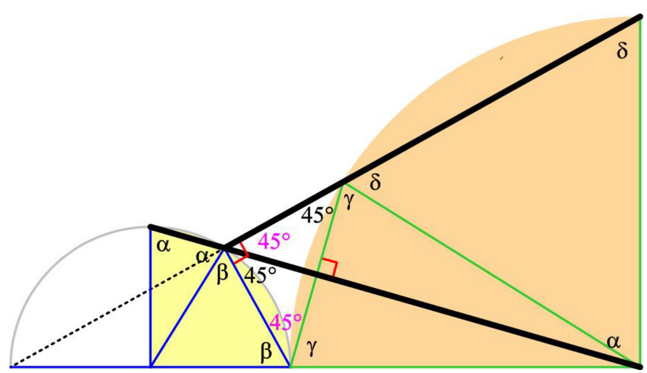


Figure 7

Figure 6 shows with $\theta = 45°$ we have two similar (blue) isosceles right triangles with a common side, and thus two congruent triangles. Therefore, α-line is a perpendicular *bisector* of the chord on the large quarter circle and so passes through the center of the quarter circle. And thus we are done.

[2] See "Existence Proofs" (https://josmfs.net/2021/09/11/existence-proofs/). One of my favorite essays.

Hidden Relations

Given θ = 45° we have all kinds of hidden relationships. Figure 7 shows one. The base of the β-triangle is perpendicular to the slant line (extended) from the large quarter circle. This means it is part of an inscribed right triangle in the semicircle determined by the small quarter circle with hypotenuse the diameter of this semicircle. So if the α-angle slant line intersects the center of the right-hand quarter circle, then the δ-slant line intersects the end of the diameter of the smaller quarter circle. Rather amazing. I tried to use this latter fact to prove θ = 45° (Figure 8), but I couldn't prove it independently of θ = 45°. This happened over and over.

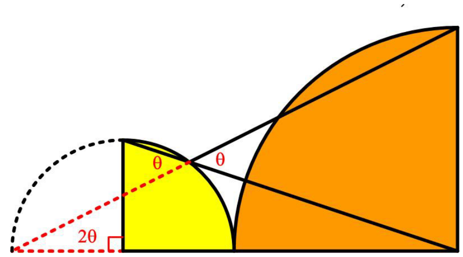
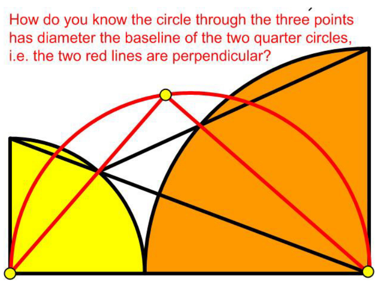
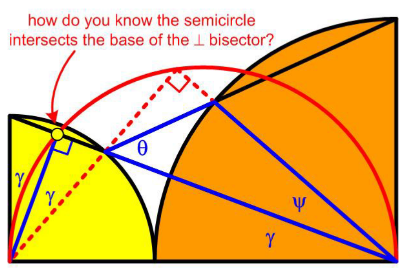


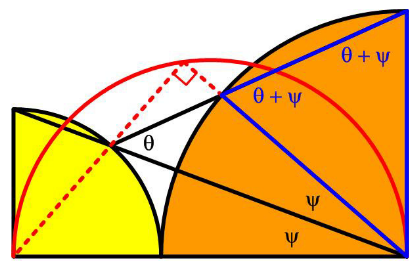
Figure 8

Here is another fact derived from θ = 45°. $\alpha + \beta = 135° = \beta + \gamma \Rightarrow \alpha = \gamma$. And $\gamma + \delta = 135° = \beta + \gamma \Rightarrow \beta = \delta$. Thus the α-triangle is similar to the γ-triangle and the β-triangle is similar to the δ-triangle. Again, I tried to prove α = γ in order to prove θ = 45°, but could not find a way.





$\psi + \gamma = 2\gamma \;\Rightarrow\; \psi = \gamma$



$2(\theta + \psi) + (90° - 2\psi) = 180°$

$2\theta = 90° \;\Rightarrow\; \theta = 45°$

Figure 9 **Figure 10** **Figure 11**

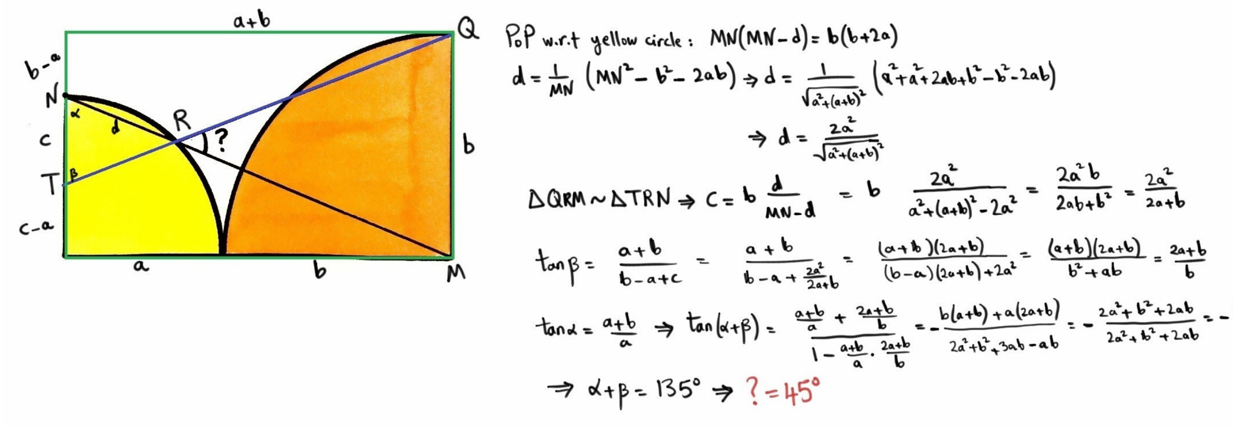
Finally, the most amazing properties shown in Figure 9 - Figure 11 were part of my initial attempts to prove θ = 45°. I extended the sides of the triangles determined by the intersections of the heavy black lines to their intersection point. I then passed a circle through these three points (Figure 9). Visio showed the radii of the two quarter circles coincided with the diameter of the circle and thus formed a semicircle (but I couldn't prove it). I then drew the altitude (perpendicular bisector) of the α-triangle (Figure 10), and it magically intersected the base on the new semicircle (but again I couldn't prove it). With these wonderful relations, I was able to prove the α-slant line bisected the right-hand angle of the right-triangle inscribed in the semicircle (Figure 10). And with this result I was able to prove θ = 45° (Figure 11). I tried many other ways to prove the α-slant line bisected the right-hand angle, but always failed. Everything seemed to depend on θ = 45°.

In fact, it seemed that proving the α-line was the angle bisecter seemed critical, and that seemed to depend crucially on it being perpendicular to the base of the γ-triangle. So that is why I ended up trying to prove ω = 90°. But the IVT is a non-geometric sledge hammer and so somewhat unsatisfactory. I would be curious to see if there is a pure geometric path that does not keep ending up requiring what it is meant to prove.

Catriona Agg Twitter Solutions

I finally checked Catriona Agg's site to see what others did. A number were quite complicated and I didn't have the energy to follow, including some arguments that approached the problem via the property shown in Figure 7 and Figure 8 above. Some fell into the trap of assuming relations they did not prove. Some quoted theorems that seemed to beg the question for me, that is, quoting a theorem seemed to hide why something was true, and the reason the theorem applied did not always seem evident?

One (https://twitter.com/Ahmed90327584)[3] hammered it with analytic geometry calculations that were quite impressive:



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[3] 16 Feb 2022 (https://twitter.com/Ahmed90327584/status/1493878529846657027)