Close Race Puzzle

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This puzzle from the Scottish Mathematical Council (SMC) Senior Mathematics Challenge ([1]) seems at first to have insufficient information to solve.

Ant and Dec had a race up a hill and back down by the same route. It was 3 miles from the start to the top of the hill. Ant got there first but was so exhausted that he had to rest for 15 minutes. While he was resting, Dec arrived and went straight back down again. Ant eventually passed Dec on the way down just half a mile before the finish.

Both ran at a steady speed uphill and downhill and, for both of them, their downhill speed was one and a half times faster than their uphill speed. Ant had bet Dec that he would beat him by at least a minute.

Did Ant win his bet?

My Solution

Figure 1 shows a space-time diagram with the information from the problem. Ant's outbound speed is v_A miles/min and Dec's outbound speed is v_D miles/min. The time intervals T_k represent critical moments in the problem. From this we get the following equations:

$$3 = v_A T_1 \quad \text{and} \quad 3 = v_D(T_1 + T_2)$$

for the outbound half of the race and

$$5/2 = 3/2 v_A T_4 \quad \text{and} \quad 5/2 = 3/2 v_D(T_3 + T_4)$$

for the inbound leg to the rendezvous. Now $T_2 + T_3 = 15$ minutes, so adding the outbound and inbound legs for each runner yields

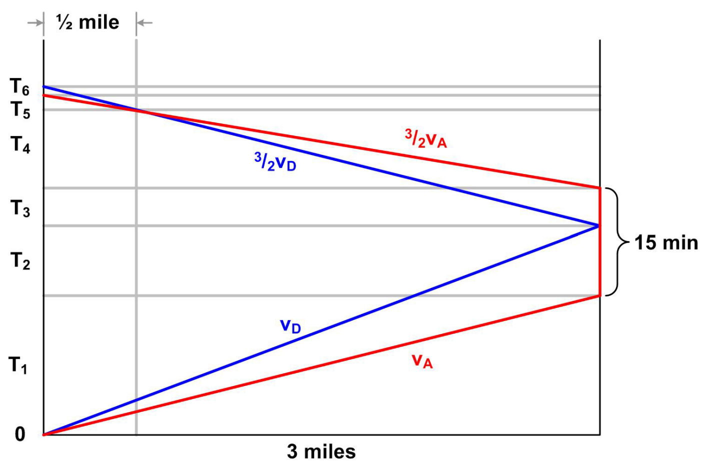

Figure 1

$$\frac{1}{v_A}\left(3+\frac{5}{3}\right)=\frac{1}{v_A}\frac{14}{3}=T_1+T_4 \quad \text{and} \quad \frac{1}{v_D}\left(3+\frac{5}{3}\right)=\frac{1}{v_D}\frac{14}{3}=T_1+15+T_4 .$$

Eliminating the times between them produces

$$\frac{14}{3}\left(\frac{1}{v_D}-\frac{1}{v_A}\right)=15$$

or

$$\boxed{\frac{1}{v_D}-\frac{1}{v_A}=\frac{45}{14}}$$

Now consider the last half mile of the race. For each runner we have

$$\frac{1}{2}=\frac{3}{2}v_A T_5 \quad \text{and} \quad \frac{1}{2}=\frac{3}{2}v_D(T_5+T_6)$$

or

$$T_5=\frac{1}{3v_A} \quad \text{and} \quad T_5+T_6=\frac{1}{3v_D}$$

So Ant arrives T_6 minutes before Dec, where

$$\boxed{T_6=\frac{1}{3}\left(\frac{1}{v_D}-\frac{1}{v_A}\right)=\frac{15}{14}>1 \text{ min}}$$

Therefore, Ant wins the bet.

SMC Solution

The SMC solution ([2]) goes directly to formulating the problem in terms of time.

Let Ant's uphill speed be a mph and Dec's be b mph. Suppose that Ant had been resting for x hours when Dec arrived (where x is between 0 and ¼). Then, calculating their times to the top of the hill and then until Ant passed Dec on the way down we have the time going up is

$$\frac{3}{a}+x=\frac{3}{b}$$

and the time going down is

$$\frac{\frac{5}{2}}{\frac{3}{2}a}+\left(\tfrac{1}{4}-x\right)=\frac{\frac{5}{2}}{\frac{3}{2}b}$$

So rearranging each of these

$$\frac{1}{b}-\frac{1}{a}=\frac{x}{3} \quad \text{and} \quad \frac{1}{b}-\frac{1}{a}=\frac{3}{5}\left(\frac{1}{4}-x\right)$$

$$\frac{x}{3}=\frac{3}{5}\left(\frac{1}{4}-x\right)$$

$$\frac{5x}{9}=\frac{1}{4}-x$$

$$20x = 9 - 36x \;\Rightarrow\; x = 9/56.$$

Dec's time for the whole race minus Ant's time for the whole race is

$$\left(\frac{3}{b}+\frac{3}{\frac{3}{2}b}\right)-\left(\frac{3}{a}+\frac{1}{4}+\frac{3}{\frac{3}{2}a}\right)=5\left(\frac{1}{b}-\frac{1}{a}\right)-\frac{1}{4}=\boxed{\frac{1}{5}\times\frac{9}{56}-\frac{1}{4}}=\frac{1}{56}>\frac{1}{60}$$

So Ant did beat Dec by more than a minute and won his bet.

SMC made an error in transcription (besides swapping d for b part way through). The last step should be

$$5\left(\frac{1}{b}-\frac{1}{a}\right)-\frac{1}{4}=\boxed{5\times\frac{3}{56}-\frac{1}{4}}=\frac{15}{56}-\frac{14}{56}=\frac{1}{56}>\frac{1}{60}\text{ hr.}$$

The SMC solution is in terms of hours and mine in terms of minutes. So

$$(1/56)\text{hr} \times 60 \text{ min/hr} = 15/14 \text{ min},$$

which is what I got.

It seems that the solvers of many of these travel problems go immediately to formulating the solution in terms of times. I have always found that a less intuitive thing to do initially. So I prefer to translate the problem statement directly into "distance equals rate times time" equations, and then solve them algebraically. Often the algebra dictates solving for the times, so the time-oriented solver's approach happens automatically.

References

[1] "Senior Division: Problems 1 S2" *Mathematical Challenge 2010−2011*, The Scottish Mathematical Council (http://www.wpr3.co.uk/MC-archive/S/S-1112-Q1.pdf)

[2] "Senior Division: Problems 1 Solutions S2" *Mathematical Challenge 2010−2011*, The Scottish Mathematical Council (http://www.wpr3.co.uk/MC-archive/S/S-1011-S1.pdf)