The Staircase Race

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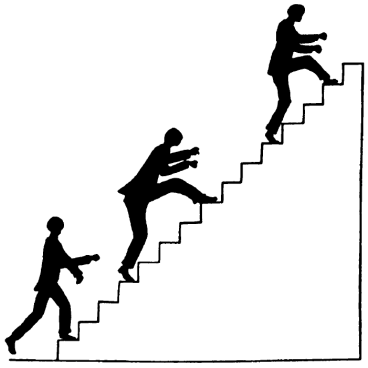
Jim Stevenson



This is a classic type of puzzle from Henry Dudeney ([1]).

This is a rough sketch of the finish of a race up a staircase in which three men took part. Ackworth, who is leading, went up three steps at a time, as arranged; Barnden, the second man, went four steps at a time, and Croft, who is last, went five at a time. Undoubtedly Ackworth wins. But the point is, how many steps are there in the stairs, counting the top landing as a step?

I have only shown the top of the stairs. There may be scores, or hundreds, of steps below the line. It was not necessary to draw them, as I only wanted to show the finish. But it is possible to tell from the evidence the fewest possible steps in that staircase. Can you do it?

My Solution

I don't know any slick way of solving this problem other than writing down the possibilities. Ackworth taking three steps at a time leaves one step to reach the top. Therefore the number of steps $N = 3n + 1$ for some integer n. Barnden taking four steps at a time leaves 3 to get to the top. So $N = 4m + 3$ for some integer m. And finally, Croft taking five steps at a time leaves 4 to get to the top, and so $N = 5r + 4$ for some integer r.

We write out possibilities for these three cases:

	r, m, n =	1	2	3	4	5	6	7	8	9
Croft	N = 5r + 4	9	14	**19**	24	29	34	39	44	49
Barnden	N = 4m + 3	7	11	15	**19**	23	27	31	35	39
Ackworth	N = 3n + 1	4	7	10	13	16	**19**	22	25	28

So the first agreement between the three runners is 19 steps. (The next agreement would occur at $79 = 19 + 3 \cdot 4 \cdot 5 = 19 + 60$ steps.)

Dudeney Solution

If the staircase were such that each man would reach the top in a certain number of full leaps, without taking a reduced number at his last leap, the smallest possible number of steps would, of course, be 60 (that is, 3 x 4 x 5). But the sketch showed us that A. taking three steps at a leap, has one odd step at the end; B. taking four at a leap, will have three only at the end; and C. taking five at a leap, will have four only at the finish. Therefore, we have to find the smallest number that, when divided by 3, leaves a remainder 1, when divided by 4 leaves 3, and when divided by 5 leaves a remainder 4. This number is 19. So there were 19 steps in all, only 4 being left out in the sketch.

References

[1] Dudeney, Henry Ernest, *536 Puzzles & Curious Problems*, (1930), Martin Gardner, ed., *Scientific American*, Charles Scribner's Sons, New York, 1967. pp.16-17