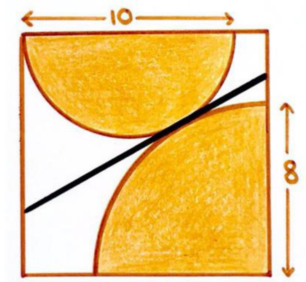
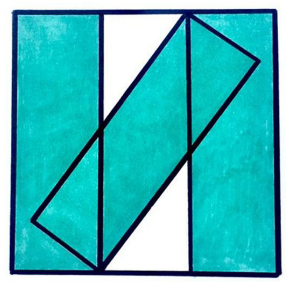
Geometric Puzzle Magnificence

2 December 2020

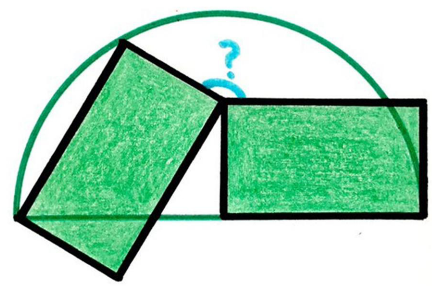
Jim Stevenson



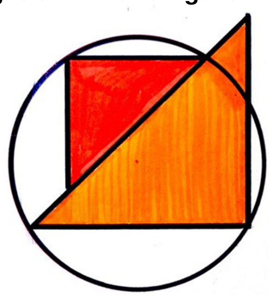
#1 A semicircle and quarter circle inside a square. How long is the black tangent line?



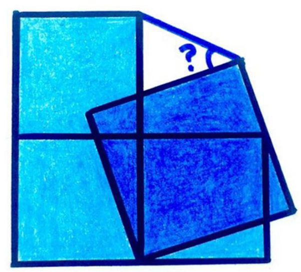
#2 The three green rectangles are congruent. What fraction of the square do they cover?



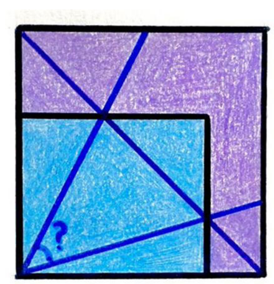
#3 The two rectangles are congruent. What's the angle?



#4 Two half squares, and a circle of radius 1. What's the total shaded area?



#5 Four squares. What's the angle?



#6 The blue rectangle covers half of the square's area. What's the angle?

Here is yet another collection of beautiful geometric problems from Catriona Agg (née Shearer). For some reason I found these a bit more challenging than the previous ones. Some of them required more time to "see" the breakthrough.

Solution to Problem #1[1]

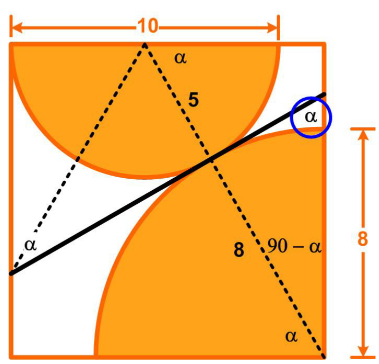


Figure 1

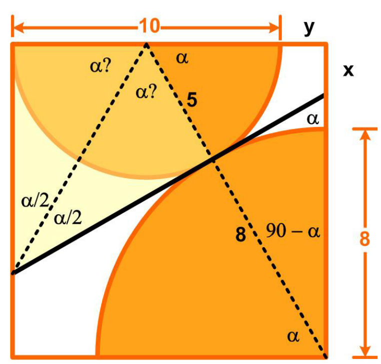


Figure 2

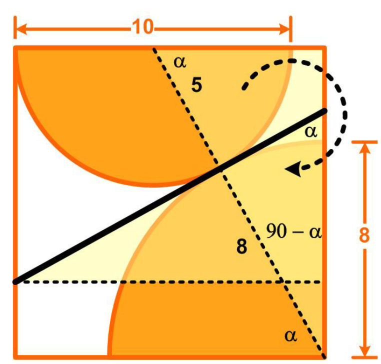


Figure 3

Since I use Viseo a lot to solve these geometric problems, I thought it might be instructive to show how Viseo led me down a blind alley. Figure 1 shows how I annotated the original drawing

[1] 5:32 AM • Aug 17, 2020 https://twitter.com/Cshearer41/status/1295292504909254658

with a line through the centers of the semicircle and quadcircle that is perpendicular to the tangent line. I also added a line from the center of the semicircle to the intersection point of the tangent line. These lines revealed two right triangles whose sides could give me the length of the tangent line. They "appeared" to be congruent, so I tried to prove that. I added the unknown angle α, and then showed where it, or its complement, appeared elsewhere. Figure 2 showed the two adjacent right triangles which were congruent. It then "looked like" α divided the semicircle into three equal angles, namely, 60°. From that I would be able to deduce the length of the tangent line.

I spent a day trying to prove geometrically that $\alpha = 60°$. Eventually I added the unknown lengths x, y to Figure 2 and could see that if $\alpha = 60°$, then the hypotenuse 13 would be twice the short leg of the right triangle, that is, I would get $5 + y = 13/2$. Equating the sides of the square and also using the Pythagorean theorem yields two equations,

$$10 + y = 8 + x$$

$$(5 + y)^2 + (8 + x)^2 = 13^2$$

Thus
$$(5 + y)^2 + (10 + y)^2 = 13^2$$

or
$$y^2 + 15y - 44 = 0.$$

So
$$y = (-15 + \sqrt{313})/2 \approx 2.7/2$$

and
$$y + 5 \approx 12.7/2$$

which is *almost* 13/2, but not quite! So $\alpha \neq 60°$, but it is close enough to fool Viseo. The moral of this story is that whatever Viseo might suggest, it still must be proved.

So back to the drawing board. I started playing with circles with diameter 13 and noticed that the tangent line seemed to be 13 long. A quick look at the diagram yielded the answer, which is shown in Figure 3. The two yellow right triangles are congruent (they have the same angles and a side in common—the edge of the square, which means the vertical right triangle can be rotated 90° to coincide with the horizontal yellow right triangle), and so have equal hypotenuses. Therefore the length of the tangent line is 13. Simple when you are not going down a seductive blind alley.

Solution to Problem #2[2]

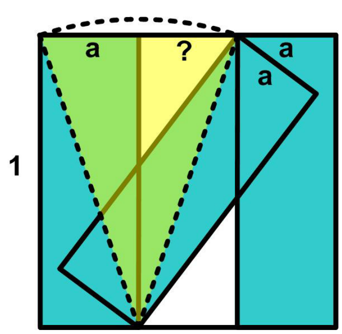


Figure 4

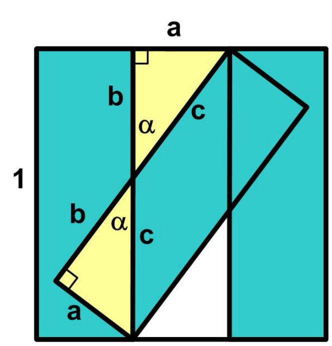


Figure 5

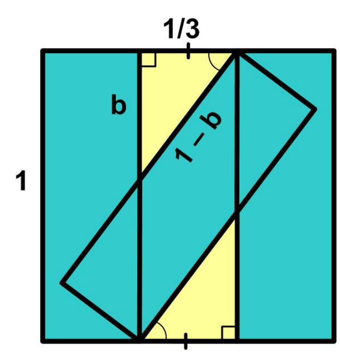


Figure 6

Since we are interested in the ratio of areas, we can assume the square has unit area and then find the fractional area of the overlapping rectangles. First, we have to show the white column is the same width as the colored rectangles. Let the width of the rectangles be a. Notice that the angled rectangle can be obtained from the left-most rectangle by pivoting it about its lower right corner until its upper left corner coincides with the upper left corner of the right-most rectangle (Figure 4). Therefore the

[2] 5:58 AM • Jul 23, 2020 https://twitter.com/Cshearer41/status/1286239151940161538

diagonal of the rotated rectangle coincides with the diagonal of the "white-column" rectangle, making a white right triangle with a side and hypotenuse equal to those of the left-hand right triangle. Thus, the two right triangles are congruent and the width of the white-column rectangle is also a, and so equal to 1/3.

Now notice in Figure 5 that the two yellow right triangles are congruent since they have two, and thus three angles in common, as well as one side (a). Similarly, in Figure 6 the two yellow right triangles are also congruent. From the Pythagorean Theorem we have

$$b^2 + 1/9 = (1 - b)^2 = 1 - 2b + b^2 \Rightarrow 2b = 8/9 \Rightarrow b = 4/9$$

Therefore, the area of the region covered by the colored rectangles is the area of the square minus the areas of the two yellow triangles, that is,

$$\text{Area} = 1 - (1/3)(4/9) = 23/27$$

Solution to Problem #3[3]

As pointed out by Simon Ellis and his sister Mary, my previous solution was flawed. I fell into the trap I try so hard to avoid, namely, I assumed what I wanted to prove, not explicitly, but rather implicitly. This is a common trap in geometric proofs. I have a skeptical gremlin in my head that tries to challenge each of the statements I make. He works best if I leave a solution for a while and then come back to it. But this time he failed.

However, the bright side is that the new solution is far more concise and closer to a Catriona Agg type of result. You just have to "see" it. Sort of like the coffin problems.

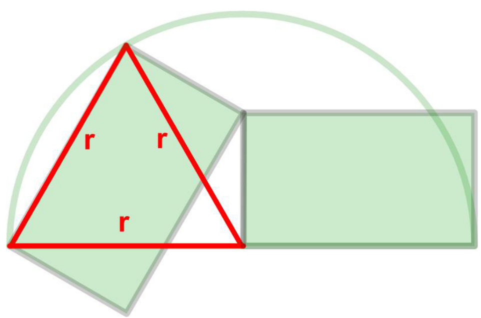


Figure 7

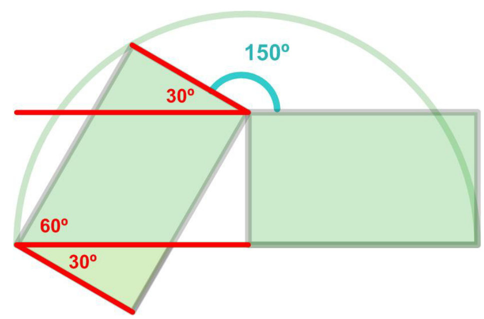


Figure 8

Since the long sides of the congruent rectangles are also the radii of the semicircle, if we add a third radius as shown in Figure 7, we obtain an equilateral triangle with vertex angle 60°. (This was the new insight.) Subtracting that from 90° leaves 30° as shown in Figure 8, which then implies the desired angle is 150°.

Solution to Problem #4[4]

Just like in Problem #3 I will choose the unspecified sizes of the half-squares to be equal (Figure 9). Since the circumscribed circle has radius 1, the edge of the resulting full square is √2, and so the area is 2. That should be the area, then, for any configuration of half-squares.

[3] 3:43 AM • Jul 29, 2020 https://twitter.com/Cshearer41/status/1288379722586521600
[4] 3:30 AM • Jul 25, 2020 https://twitter.com/Cshearer41/status/1286926863344992258

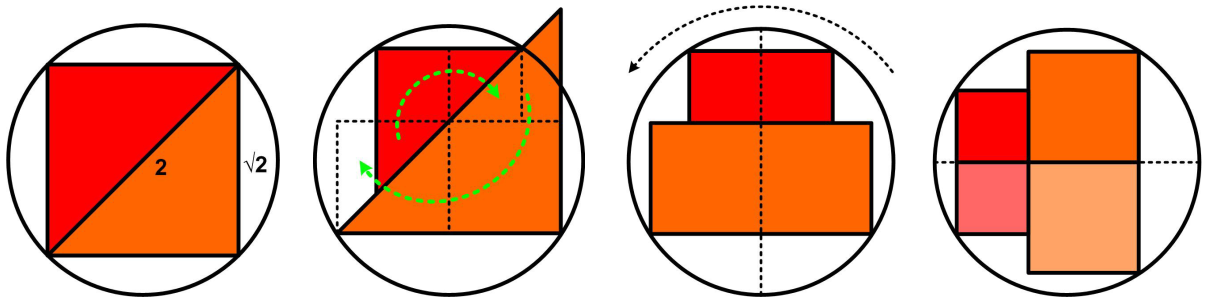


Figure 9 **Figure 10** **Figure 11** **Figure 12**

Figure 10 shows the general situation. I have bisected the edges of the half-squares as shown. Then by rearranging the resulting congruent triangles in the half-squares as indicated, we end up with two rectangles composed of two squares each (Figure 11), that is, the diameter of the circle cuts the rectangles into pairs of equal squares. Now rotate the figure counter clockwise 90° to obtain Figure 12. Then we have two semicircles containing the same two squares as given in Presh Talwalkar's Sum of Squares Puzzle.[5] In that puzzle the sides of the two squares were x and y, and the sum of their areas was $x^2 + y^2 = r^2$, where $r = 10$. In our puzzle, the radius $r = 1$, so the sum of the areas of the squares in the semicircle is 1. But we have two semicircles, so the total area is 2, which is just what we predicted.

We followed the time-tested math principle of reducing the problem to one we already know how to solve.

I checked Catriona's site and I did not see solutions like mine.

Solution to Problem #5[6]

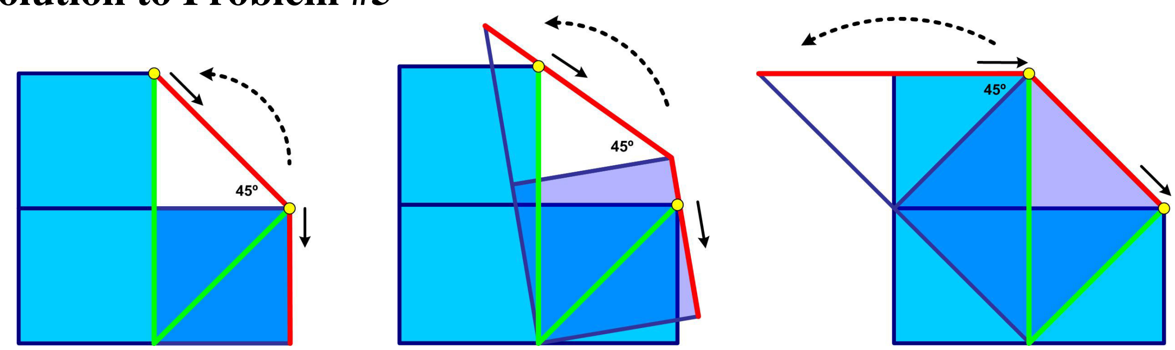


Figure 13 **Figure 14** **Figure 15**

So again the position and size of the dark blue square was not specified, so I chose the simplest, which was equal to one of the light blue squares as shown in Figure 13. That meant the desired angle was 45°. Again I will show the position and size of the dark blue square does not matter.

Figure 14 and Figure 15 show the interesting result that as the dark blue square rotates and enlarges so that its edge continues to touch the corner of the lower right light blue square, the upper red line keeps passing through the upper right corner of the upper left blue square.

This reminded me of Problem #4 in Catriona Agg's previous collection, Geometric Puzzle Munificence,[7] and the Curve Making Puzzle.[8] Figure 16 makes this clear by changing the point of

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5 http://josmfs.net/2020/09/05/sum-of-squares-puzzle/
6 4:21 AM • Aug 14, 2020 https://twitter.com/Cshearer41/status/1294187497229033472
7 https://josmfs.net/2020/11/28/geometric-puzzle-marvels/
8 http://josmfs.net/2020/11/07/curve-making-puzzle/

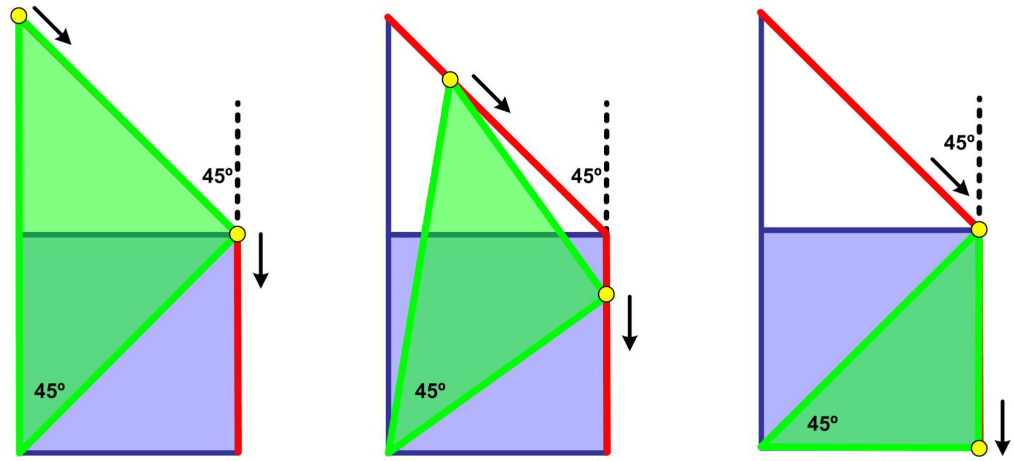

Figure 16

view from having the dark blue square move as shown in Figure 13 - Figure 15 to fixing its position and letting the light green lines slide along the red line. The green lines define a set of similar triangles. As the endpoint of the right-hand-leg of the triangle slides down the vertical edge of the blue square, the endpoint of the left-hand-leg of the triangle slides along a similar curve, namely, a straight line. Since the left-hand-leg is tilted 45° with respect to the right-hand-leg, the line it traces will be tilted by 45° with respect to the line traced by the right-hand-leg.

When I checked the solutions at Catriona's Twitter site, no one used this argument. Most solutions claimed the upper right vertex of the dark square traced out the arc of a circle, or equivalently claimed that the relevant vertices were cyclic, that is, lay along a circle. The justification for this seemed a bit obscure to me and relied on properties of cyclic points that I was not familiar with.

Solution to Problem #6[9]

Aside from my detour in Problem #1, this problem turned out to be the most challenging.

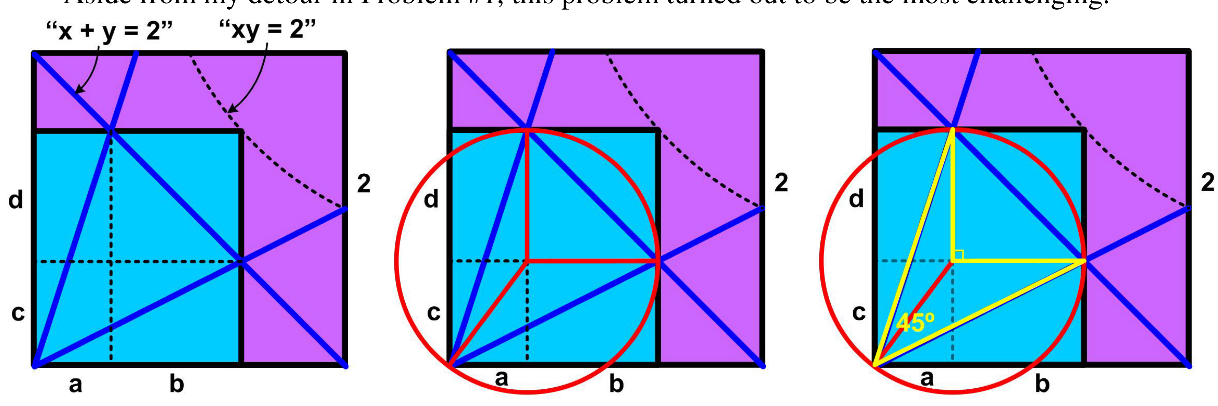

Figure 17 **Figure 18** **Figure 19**

For convenience we assume the square has edges of length 2. Then its area will be 4 and that of the blue rectangle half that, or 2. Figure 17 shows the original problem statement with some (dashed) lines added, along with some annotations. The dashed curve in the upper right corner represents a hyperbola of the form xy = 2. So any rectangle with area 2 will have its upper right corner lying on the hyperbola. Secondly, the diagonal blue line passing from one corner of the square down to the

[9] 7:36 AM • Jul 26, 2020 https://twitter.com/Cshearer41/status/1287350989536866305

other has an equation of the form x + y = 2. These statements yield the following equations in the designated variables.

$$(a + b) + c = 2$$

$$a + (c + d) = 2$$

Therefore, b – d = 0, or b = d, which means the upper right region of the blue rectangle is a square with sides b. Furthermore, a + c = 2 – b. From the area constraint we have

$$2 = (a + b)(c + d) = ac + b(a + c) + b^2 = ac + b(2 – b) + b^2 = ac + 2b$$

So

$$ac = 2 – 2b \tag{*}$$

Now we draw a circle of radius b, centered as shown in Figure 18, and passing through the two points of intersection of the blue lines with the blue diagonal. The figure shows the circle also passes through the lower left corner of the square, but we must prove that. This is equivalent of showing $a^2 + c^2 = b^2$. We need expressions involving the squares of these variables.

$$a + c = 2 – b \;\Rightarrow\; (a + c)^2 = (2 – b)^2 \;\Rightarrow\; a^2 + 2ac + c^2 = 4 – 4b + b^2$$

From this result and equation (*) we get

$$a^2 + (4 – 4b) + c^2 = 4 – 4b + b^2 \;\Rightarrow\; a^2 + c^2 = b^2$$

which is what we wanted to show.

Therefore Figure 19 obtains and we see that the unknown angle we are searching for is the inscribed angle of the red circle with central angle of 90°. Hence our desired angle is half of that, or 45°.

Comment. Finding the critical red circle was serendipitous—just playing around in Viseo. I looked at a related circle and was quite surprised by its properties (Figure 20). The new circle is the old circle reflected about the diagonal of the square. Its center will be the diagonally opposite corner of the small square region in the blue rectangle. Since the original circle passed through a vertex of the large square, so will the reflected version. But what is really startling is that the intersections of this new circle with the large square are also the points of intersection of the blue lines with the edges of the square. I haven't bothered to prove it, but it looks to be true.

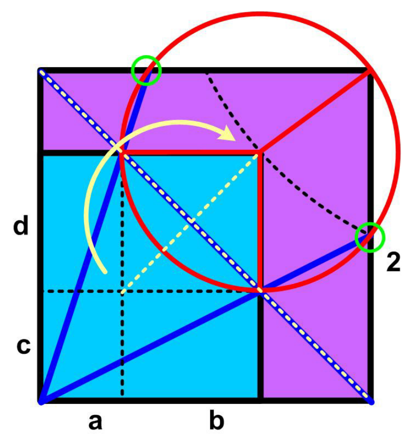

Figure 20