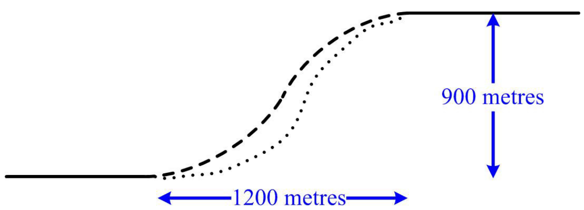
Road Construction Problem

14 July 2022

Jim Stevenson



This is an interesting problem from the Scottish Mathematics Council (SMC) 2014 Senior Math Challenge ([1]).

Two straight sections of a road, each running from east to west, and located as shown, are to be joined smoothly by a new roadway consisting of arcs of two circles of equal radius. The existing roads are to be tangents at the joins and the arcs themselves are to have a common tangent where they meet. Find the length of the radius of these arcs.

My Solution

Figure 1 represents a parametrization of the problem. The miss-alignment of the circles with respect to one another is represented by the variable x, where x may be positive, negative, or zero. The radius of the equal circles is given by r. Since the circles are tangent to one another, the distance between their centers is $2r$. So we have the following relationships.

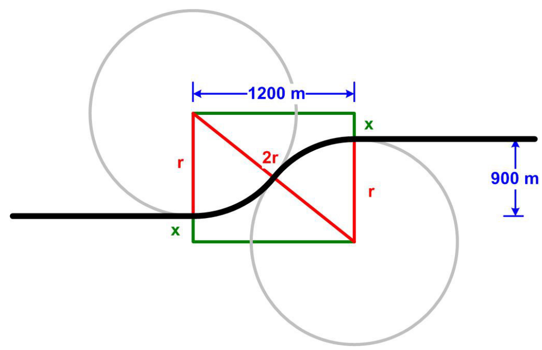


Figure 1

$$r - x = 900$$

and $$(r + x)^2 + 1200^2 = 4r^2$$

Eliminating x yields $$(2r - 900)^2 + 1200^2 = 4r^2$$

or $$4r^2 - 12\cdot 300r + (3\cdot 300)^2 + (4\cdot 300)^2 = 4r^2$$

or $$12\cdot 300r = (5\cdot 300)^2$$

or $$r = 25\cdot 300/12 = 25\cdot 50/2 = 625 \text{ meters}$$

SMC Solution

SMC provided a slick solution ([2]). From Figure 2,

$$\cos x = \frac{900}{1500} = \frac{3}{5}$$

$$\cos x = \frac{1500}{4}\frac{1}{r} = \frac{375}{r} \qquad (*)$$ [1]

$$r = 375 \times \frac{5}{3} = 125 \times 5 = 625$$

i.e. the radius of the circles is 625 metres.

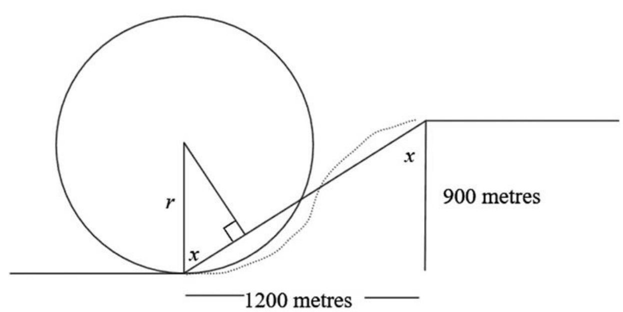


Figure 2

[1] JOS: It took me a while to understand this step. See Comment 1 below.

Comment 1. First, the SMC solution assumes the line joining the endpoints of the straight roads goes through the point of tangency of the circles. This is not immediately evident, at least to me.

Figure 3 shows the same setup as in Figure 1, but with the blue lines added. These two lines join the point of tangency with each of the endpoints of the straight roads. We want to show they form a single straight line. The vertical radii are parallel, so the blue isosceles triangles have the same vertex angle β, since the line joining these vertices is a straight line. Therefore the triangles are congruent and their other angles are equal to the same angle α (= x). Since the line joining the centers of the circle is a single straight line, $\theta + \alpha = 180°$. But that means that the blue line segments also lie on a straight line, and we are done.[2]

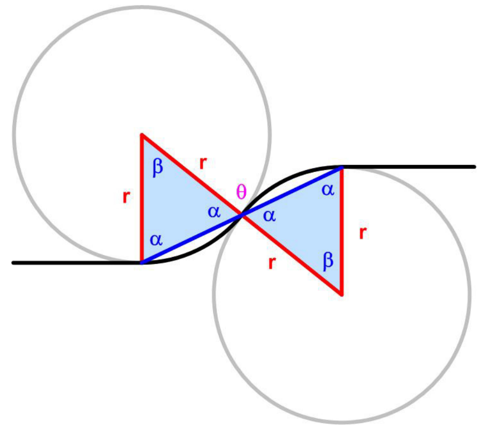


Figure 3

Now, with this established, the SMC solution drops a perpendicular from the center of the circle to the (blue) chord, which is then a bisector. The same thing happens with the other circle. Thus the blue line (of length 1500 m) is divided into 4 equal segments, each of which equals $r \cos \alpha = r \cos x$. This yields equation (*) above.

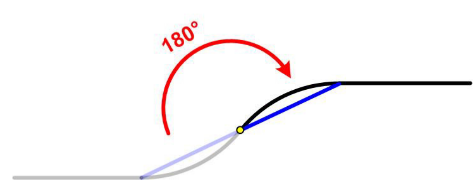
Comment 2. Of course we know from my post "Train Wreck Puzzle"[3] that this road construction does not provide a "smooth" join in either the calculus sense of in infinitely differentiable curve or the physical sense of a gradual change in direction. The change from the zero curvature of the straight highway to the constant curvature of the arc of the circle will cause an impulse force on the vehicle to make it turn. And the switch from one circular arc to the other will cause an impulse force of twice the amount as the curvature switches from positive to negative. Even if the car stays on the road, the passengers will be tossed abruptly back and forth. Still, it was a fun problem.

References

[1] "Senior Division: Problems 2 S4" *Mathematical Challenge 2013–2014*, The Scottish Mathematical Council (http://www.wpr3.co.uk/MC-archive/S/S-1314-Q2.pdf)

[2] "Senior Division: Problems 2 Solutions S4" *Mathematical Challenge 2013–2014*, The Scottish Mathematical Council (http://www.wpr3.co.uk/MC-archive/S/S-1314-S2.pdf)

2 (10/22/2022) Actually, there is a faster way to see this. By the symmetry of the road configuration, if a chord is drawn from the beginning of the first circular curve to the tangent point of the second, then a rotation of 180° around the tangent point will take the first segment of the road onto the second segment, thus producing a line through the tangent point.



3 https://josmfs.net/2021/03/06/train-wreck-puzzle/