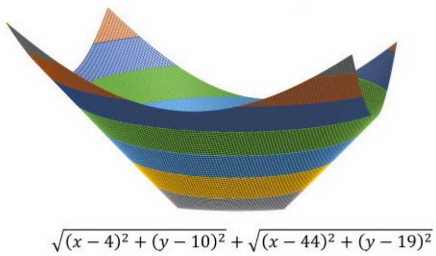
# Square Root Minimum

Jim Stevenson

8 June 2022




This seemingly impossible problem from Presh Talwalkar turned out to be quite solvable upon reflection ([1]).

A similar question was given to students in Thailand. For real numbers $x$, $y$, what is the minimum value of

$$\sqrt{((x-4)^2 + (y-10)^2)} + \sqrt{((x-44)^2 + (y-19)^2)}$$

## My Solution

When I first confronted this problem, I thought of finding the values ($x$, $y$) that would provide the minimum via an algebraic approach that would require removing the radicals. But this would elevate the complexity of the problem to polynomial expressions involving quartic terms.

Then I started thinking geometrically. The sum of the radicals became the sum of the distances from a point ($x$, $y$) in the plane to two fixed points (4, 10) and (44, 19). The locus of all such points that gave the same value is an ellipse with the two fixed points as foci. This value would be twice the length of the semimajor axis $a$ of the ellipse or the length of the major axis $2a$ (Figure 1). Clearly the

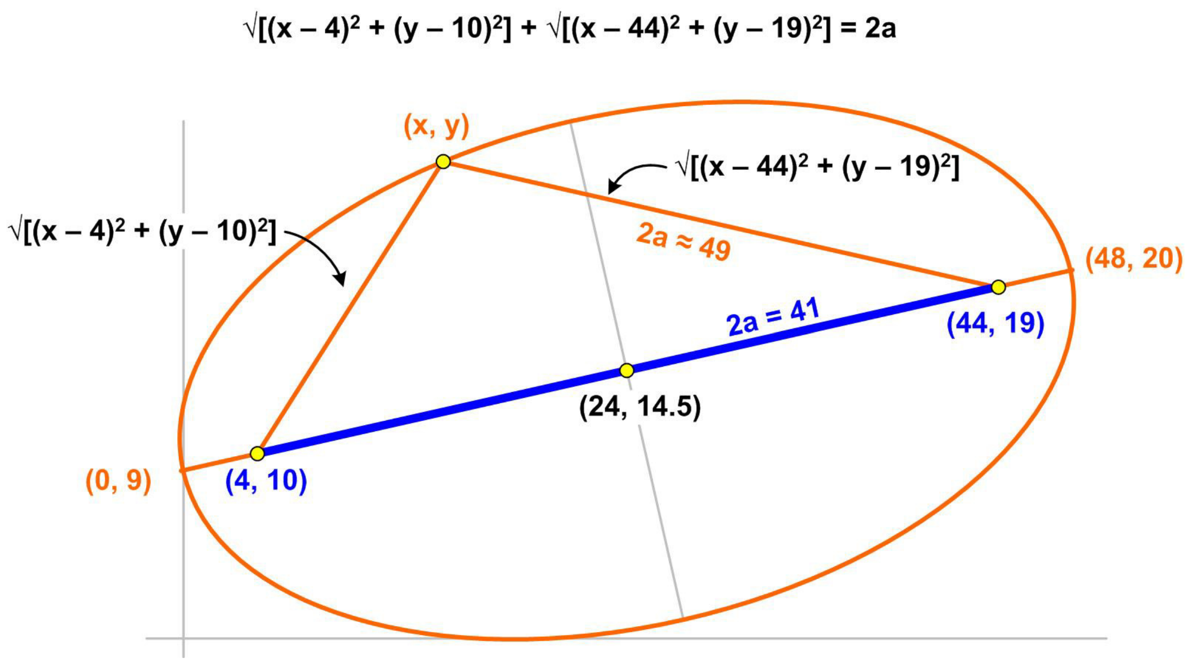



**Figure 1**

path with the shortest distance between the two foci would be a straight line, with length given by

$$\sqrt{[(44-4)^2 + (19-10)^2]} = \sqrt{(40^2 + 9^2)} = \sqrt{1681} = 41.$$

Notice that this line is also the limiting "ellipse" as the semiminor axis shrinks to zero and the sum of the lengths to the two foci (major axis) approaches a minimum, the line joining the foci. Furthermore, any point ($x$, $y$) along the line between the two foci will produce the same sum of distances to the two foci, and so there is no single point producing a minimum value, but rather a line of points.

Also, we can think of the ellipses as representing constant sum-of-distances contours, whose values decrease as the ellipses shrink, until they converge on the minimum contour, which is the straight line between the foci (Figure 2).

A three-dimensional image of the surface represented by these contours is shown in Talwalkar's figure (Figure 3).

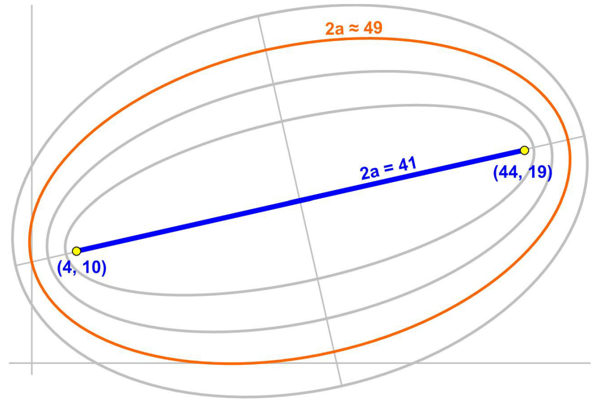



**Figure 2**

# Talwalkar Solution

Talwalkar's solution ends up with basically the same argument and calculation that I used.

One way to approach the problem is to graph

$$z = \sqrt{((x-4)^2 + (y-10)^2)} + \sqrt{((x-44)^2 + (y-19)^2)}$$

and see where the minimum value is. The graph resembles a parabaloid, and it appears the minimum value occurs at $z = 41$.

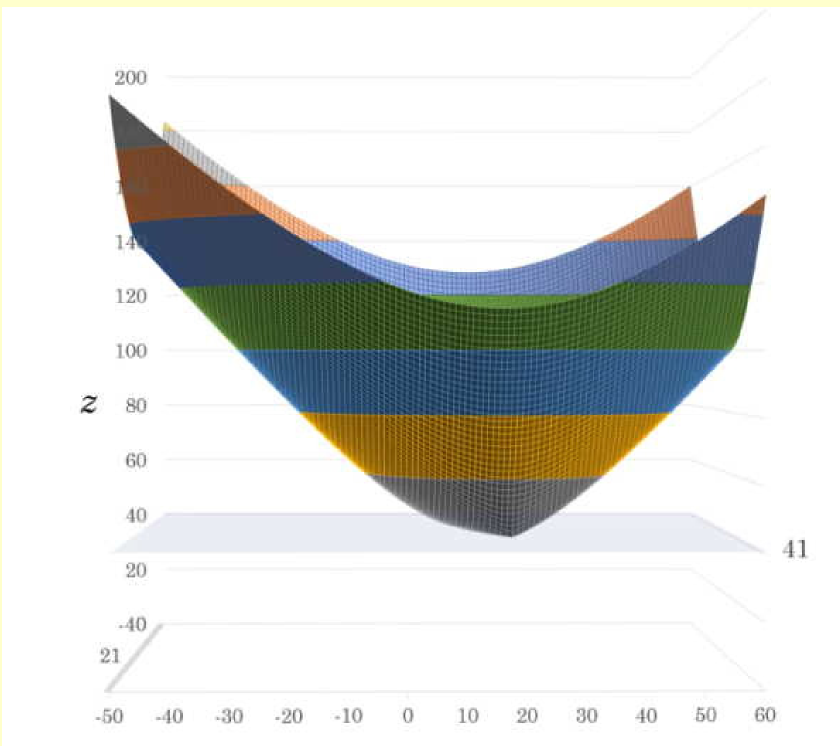
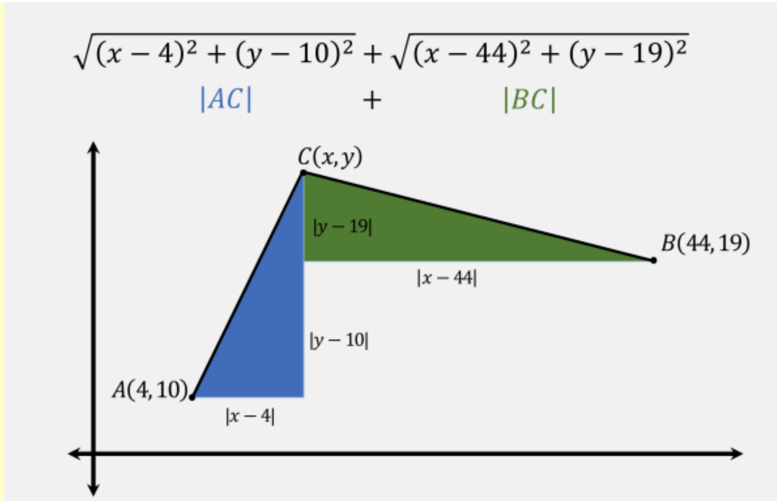



**Figure 3**

While this approach gives a sense of the answer, it does not reveal the structure of the question. There is actually an elegant way to solve this problem involving geometry!

Consider a plot in the *x*-*y* plane, and let $A = (4, 10)$, $B = (44, 19)$, and $C = (x, y)$.




By the distance formula:

$$|AC| = \sqrt{((x-4)^2 + (y-10)^2)}$$
$$|BC| = \sqrt{((x-44)^2 + (y-19)^2)}$$

Then by the triangle inequality we have:

$$|AC| + |BC| \geq |AB|$$

Equality holds exactly when *C* is along the line segment *AB*, and the minimum value is the length of *AB*. We again use the distance formula to get:

$$|AB| = \sqrt{((44-4)^2 + (19-10)^2)} = \sqrt{(40^2 + 9^2)} = 41$$

Thus the minimum value is equal to 41.


**Special thanks this month to:**

Kyle

Lee Redden

Mike Robertson

Daniel Lewis

Thanks to all supporters on Patreon!


**References**


https://math.stackexchange.com/questions/1776867/what-is-the-minimum-value-of-a-radical-sum

https://math.stackexchange.com/questions/1307398/find-the-minimum-value-of-sqrtx24-sqrty29?rq=1


# References


[1] Talwalkar, Presh, "Minimum of Sum of Square Roots," *Mind Your Decisions*, 7 June 2022. (https://mindyourdecisions.com/blog/2022/06/07/minimum-of-sum-of-square-roots/)