Two and a Half Circles

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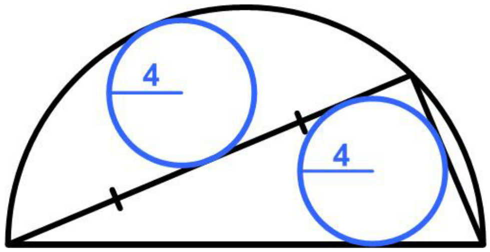

Here is a problem from the 2022 Math Calendar ([1]).

Two small circles of radius 4 are inscribed in a large semicircle as shown. Find the radius of the large semicircle.

As before, recall that all the answers are integer days of the month.

As seemed to be implied by the original Math Calendar diagram, I made explicit that the upper circle was tangent to the midpoint of the chord. Otherwise, the problem is insufficiently constrained.

Solution

Figure 1 shows my parameterization of the problem and steps in the solution. The radii of the small blue circles are r = 4 and the radius of the semicircle is R.

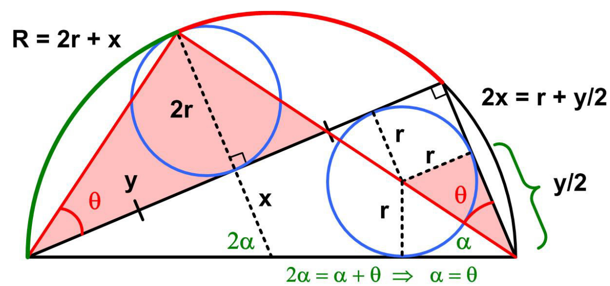

Figure 1

Drop a perpendicular to the chord from the tangent point of the top circle down to the semicircle. By symmetry it will bisect the chord of the semicircle and thus pass through its center. Therefore it is a radius of the semicircle. Let x be the distance along that radius from the lower tangent point to the center, so R = 2r + x, and let y be half the length of the chord.

Join one end of the diameter of the semicircle to the upper tangent point of the upper blue circle, and then join this point to the other end of the diameter (red line). The two inscribed angles of the semicircle thus formed subtend the same arc and so are equal and designated θ. Since the red triangles are right triangles with one equal angle, they are similar. Let α be the angle shown in the diagram. Then it forms an inscribed angle of the semicircle with central angle 2α. Since the relevant lines are perpendicular to the same line, they are parallel, which implies $2\alpha = \alpha + \theta$. Thus $\alpha = \theta$, which means the red line passes through the center of the lower blue circle. So

$$2r / y = \tan \theta = r / (y/2) \quad \Rightarrow \quad 2x = r + y/2$$

Therefore

$$x = R - 2r \quad \text{and} \quad y = 4R - 10r$$

means

$$R^2 = (R - 2r)^2 + (4R - 10r)^2$$

Solving the resulting quadratic equation yields

$$R = (13/4)r \ \text{ or } \ 2r$$

R = 2r implies x = 0, a contradiction, so R = (13/4)r = (13/4)·4 = 13.

References

[1] Rapoport, Rebecca and Dean Chung, *Mathematics 2022: Your Daily epsilon of Math*, Rock Point, Quarto Publishing Group, New York, 2022. July