Symbolic Algebra Timelines

26 July 2022

Jim Stevenson

As I am sure is common with most mathematicians, I had become interested in the history of the development of mathematical symbols, first for numbers (numerals) and then for algebra (symbolic algebra). Joseph Mazur's book *Enlightening Symbols* ([1]) provided an excellent history of this evolution. His focus on the development and significance of symbolic algebra in the Renaissance was especially illuminating. I also augmented Mazur's information with details from Albrecht Heeffer's work (e.g. [2]).

Such a subject cries out for a timeline to appreciate the order and timing of discoveries, which Mazur provided, concentrating on the Renaissance. I decided to both simplify Mazur's version and expand it to cover the evolution of numbers and their notation, as well as to set the whole enterprise in the context of historical periods (Figure 1). I relied further on Kline's 1972 history of math ([3]). My designation of historical periods represents the Western bias of my formal education, but I can't think of a viable alternative at this late stage. The boundaries of these periods are fuzzy, or one might even say fractal, in that they often varied geographically. Furthermore the periods are meant to be chunky for ease of recall and to avoid the controversies of multiple experts.

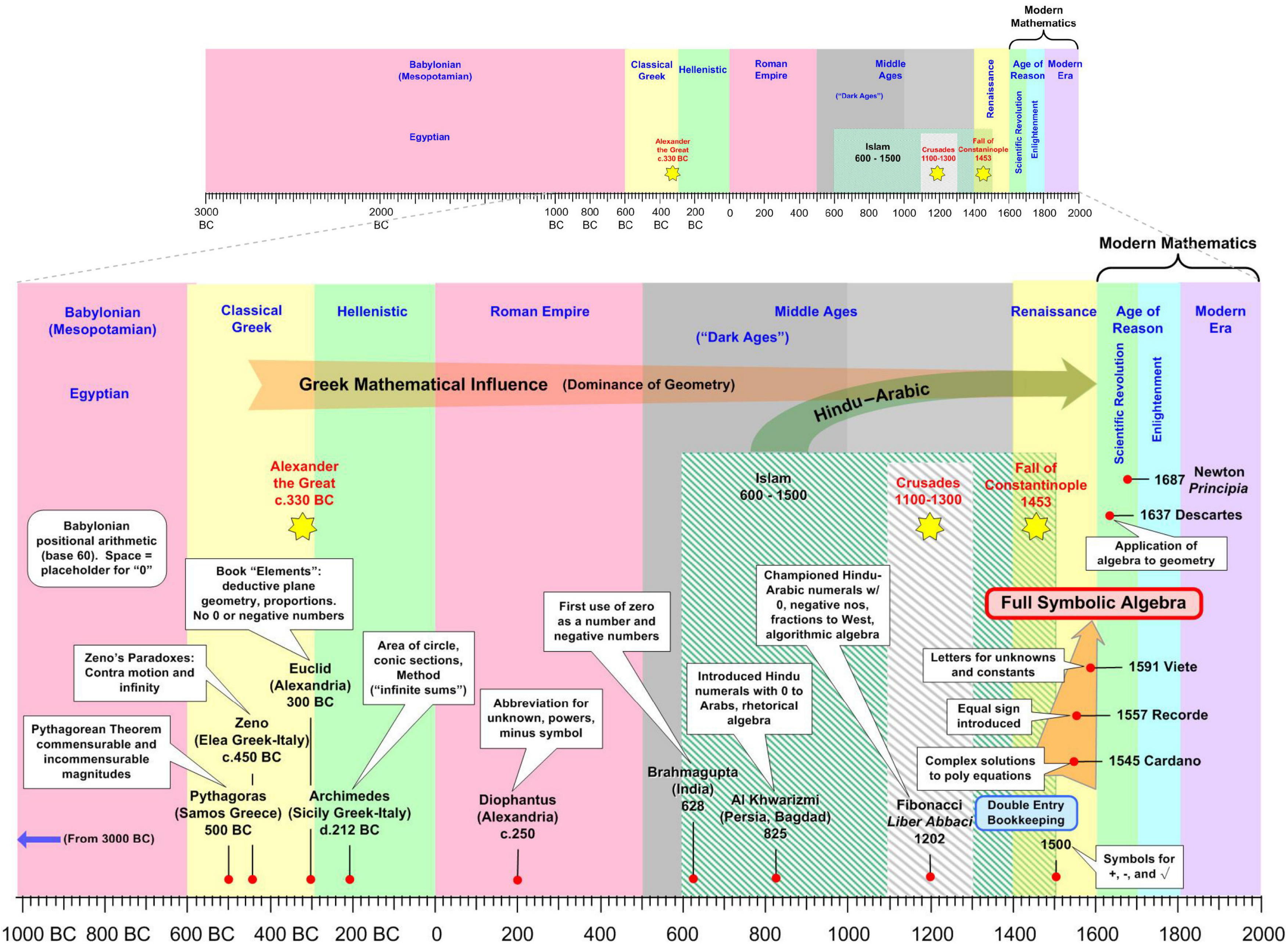
At the heart of such a presentation is the question of sources and evidence, as well as who influenced whom versus the independence of discoveries. This is a massive scholarly tangle beset on all sides by controversy. Cultural prejudice often clouds objective assessment of the frequently vague and conflicting evidence. For example, for years Western bias assumed that all mathematics basically derived from the Greeks, even if it detoured through the Arabs. There certainly was cross-fertilization between West and East, especially with astronomy, but we now know that pivotal discoveries came from the East, such as the *number* zero and negative numbers. (For my crude purposes I consider the "West" to refer mainly to Latin, largely Roman Catholic, western Europe, and the "East" to refer mainly to Greek, Byzantine, Orthodox Christian eastern Europe, followed by the multi-religious Middle East (Syria, Iran/Persia), South Asia (India), and East Asia (China).)

Hopefully my timeline reveals some important historical features:

Common School Math is *Very* Recent

Knowledge of mathematics seems to have existed for over 5000 years, more or less coincident with the advent of writing. It may go back earlier than 3000 BC to perhaps 3500 BC, depending on the dating and interpretation of some artifacts. But our use of symbolic algebra is only about 400 years old—8% of the math timeline. And we have only had the modern era of mathematics for about half that time. Even though the first recorded discussion of the number zero and negative numbers that we know about was by Brahmagupta around 600 AD in India, it evidently took another 200 years for these ideas to reach the Arabs via Al-Khwarizmi in Baghdad (825 AD), and then another 400 years for them to migrate to Italy in the West via Fibonacci (1202 AD). (There is evidence that others were aware of these ideas in Fibonacci's time, but his *Liber Abaci*, like Euclid's *Elements*, collected this knowledge in a most comprehensive and enduring way.)

More than just zero and negative numbers was transmitted in 1202, however. There was also the marvelous positional decimal notation from India that greatly simplified the expression of numbers. But these advances still languished from common use for another 200 years (largely due to the lack of a cheap, disposable medium on which to carry out the computations, i.e., paper, so that most reckoning continued to be done with the "hand calculators" of the time, namely, the abacus.) Thus the use of the advanced notion of numbers almost had to wait for the arrival of symbolic algebra.



Mazur, *Enlightening Symbols* (2014); Kline, *Mathematical Thought from Ancient to Modern Times* (1972); Heeffer, "On the Nature and Origin of Algebraic Symbolism" (2009)

Figure 1 Augmented Symbolic Algebra Timeline

But even then there was a reluctance to use negative numbers, and so equations that required negative coefficients were written separately with the negative terms moved to the other side of the equals sign to render them positive. There is some chicken-and-the-egg debate about the relationship between the coincidentally arising double-entry bookkeeping and the advent of negative numbers in the West. The fact that double-entry bookkeeping also avoids negative numbers with its "positive" terms on both sides of the balance sheet suggests to me that it does not represent an acceptance or use of negative numbers, but rather an avoidance.

Historical significance for learning math. What this lengthy mathematical gestation suggests is that for humans it is intuitively easy to understand counting numbers (1, 2, 3, …) and how to add and multiply them, but it is conceptually rather difficult to augment them with a number zero, and even more difficult to include negative numbers and their behavior. The idea of incorporating fractions (or ratios of counting numbers) as *numbers* adds to the difficulties. So the challenge of elementary school teachers teaching basic arithmetic is considerable and should not be underestimated. It was not until the middle of the 19^{th} century with Hamilton's abstract quaternion numbers that the essential structure of a "number system" became clear. It was given the name "field" and defined by a set of abstract properties or "rules".[1] But this "explanation" takes even more understanding and sophistication, and so is probably out of reach of ordinary folk who are just trying navigate the often perplexing manipulation of "numbers."

Legacy of Alexander the Great

The impact of Alexander the Great on history is something that every educated person should be aware of. The designation of Greek culture (Hellenism) after Alexander is given as "Hellenistic." It is meant to capture the blending of Greek ideas with those of his conquered territories, in particular, with ideas flowing from his Eastern conquests. We have seen one example of this in my post "Greek-Indian Connection"[2] where Greek artistic traditions influenced the representations of the Buddha in India. But influence also flowed westward. The Seleucid Empire (311-63 BC) became a conduit of eastern Zoroastrian ideas of the duality of good and evil, (becoming personified in gods and devils, angels and demons), as well as the notion of an apocalypse and last judgment (e.g.[4]).[3] This period corresponds to the gap in the Protestant Christian Bible between the Old and New Testaments which is filled with the non-canonical, apocryphal books that are included in the Roman Catholic and Orthodox Bibles. What happens in this period is not entirely clear, but a changed Judaism emerges that supports an apocalyptic Jesus prophesizing the imminent Endtime (Kingdom of God). Philip Jenkins described the second century BC as the "missing century" ([5]). Important things happened, but we don't have the records and sources to make it clear.

The Hellenistic period did not end with the rise of the Roman Empire, but continued, especially in the eastern part of the empire, and in particular in Alexander's city in Egypt. Alexandria had a profound impact on the development of mathematics. Euclid produced his compendium *The Elements* there about 300 BC (oldest extant Greek copy 888 AD ([1])), followed by Diophantus (c.250 AD) and Pappus (c.300 AD). And of course there were the great astronomical and geographical works of the Alexandrian mathematician Ptolemy, *The Almagest* and *Geography* (c. 150 AD), that were so popular that they were copied innumerable times and so survived. But other great Greek mathematicians lived in this period, such as the third century BC geometer Apollonius of Perga, a Hellenized city in Anatolia (Asia Minor), who later resided in Alexandria and developed the names and further properties of the conic sections (ellipses, parabolas, hyperbolas). And then there was the greatest of all ancient mathematicians, Archimedes of Syracuse, a city in the Greek colony of

1 https://en.wikipedia.org/wiki/Field_(mathematics)
2 https://josmfs.net/2021/10/23/greek-indian-connection-alexanders-legacy/
3 https://en.wikipedia.org/wiki/Zoroastrianism

Sicily (Italy), also living in the third century BC. He computed the areas and volumes of numerous geometric shapes with curved boundaries and even prefigured the integral calculus with his ever-diminishing "infinitesimal" approximations—a method only resurrected some 2000 years later by Kepler during his efforts to establish the "equal areas" law.[4]

Transfer of Greek Knowledge—"Dark Ages"

Controversy. The Dark Ages is currently a maligned designation that from my admittedly inexpert viewpoint still has merit, at least as I interpret it. For me it emphasizes the difference between what was happening in the West and what was happening in the East (as I defined them).

The whole of the Middle Ages covers the approximate millennium from the end of the Roman Empire to the beginning of the Renaissance, roughly 500 – 1400 AD. The designation Renaissance ("rebirth") has been used to signify the re-entry of Greek knowledge into Western Europe, ostensibly via the Muslim communities. There were actually several "renaissances" beginning around 1100 in the Late Middle Ages and occurring in different regions of Europe at different times ([7]). For example, the Muslim philosopher ibn Rushd (Averroës) introduced Aristotelianism to Europe via Spain in the 12th century. His works became the basis a hundred years later for Thomas Aquinas's merging of faith and reason in his monumental *Summa Theologica*.

But the first half of the Middle Ages, roughly 500-1000 AD, is a veritable wasteland for science and especially math in the Latin, Roman Catholic Western Europe, which is why I still think the Dark Ages is a suitable term in that regard. This is especially clear when one compares the prolific and enduring creations of the mathematicians in the Greek and Hellenistic eras that precede this period and the explosion of advances in the Renaissance and Scientific Revolution that follow this period, as well as the burgeoning of Arabic science during this period after exposure to the Greek and Indian ideas. Revisionist historians of science, such as the vituperative Thony Christie, strongly demur and denigrate anyone who thinks otherwise ([8], [9], and numerous other posts). But perhaps a mitigating factor is that Christie is largely challenging the negative association of the Roman Catholic church with this phrase and arguing that learning persisted in church institutions, such as monasteries, which were repositories of ancient documents. To my naïve viewpoint this *passive* preservation of ancient wisdom is in stark contrast with the *active* engagement of the Muslim world with the Greek knowledge they found when they translated it and combined it with ideas from India to advance mathematics and astronomy. But I would like to leave this debate for another day.

Transfer of Greek knowledge. I plan to write a more detailed post about the early transfer of Greek knowledge, which is less known than the later transfer of Arabic translations in the Late Middle Ages and the Renaissance, and is utterly fascinating.

For now, the main point is that any survival of ancient records is a miracle. The ravages of history occurred not just with weather, wars, and religious bigotry, but also through the evolution of media on which to record the knowledge. From clay tablets through awkward, fragile papyrus scrolls, to parchment (animal skins) codices (books), and then paper, together with changes in

[4] Archimedes did not actually use the later controversial term "infinitesimal", but did use the sum of areas of an ever-increasing number of shrinking simpler geometric figures to approximate the area of a curved geometric region. He established the "limit" via the exhaustion method. It should be noted, however, that Archimedes did use a heuristic procedure with a strong affinity to "infinitesimals" or "indivisibles" called "The Method of Mechanical Theorems" that was only discovered in 1906 in a Medieval palimpsest. The method used the principle of the lever arm to balance the "indivisibles" dissection of the figure whose area was to be measured with the "indivisibles" of a corresponding rectangle on the other side of the fulcrum. This method produced the relationships that he would later establish more rigorously via the exhaustion method. (See *Wikipedia*: https://en.wikipedia.org/wiki/The_Method_of_Mechanical_Theorems and [6])

languages and script, the transformations required constant copying in order for records not to become obsolete and lost.

Until the advent of the moveable type printing press of Gutenburg in the mid-1400s, such copying efforts were laborious and fraught with error. For example, there is no *original* copy of the Bible; all versions are copies of multiple copies. This state of affairs has provided scholars centuries of frustrated debate as to what is the "true" Bible. But at least the "popularity" (!) of the Bible meant it was copied and translated over and over. Such was not the destiny of most other documents and especially in the area of abstruse mathematics (save for Euclid and Ptolemy).

Archimedes' more accessible earlier Greek works were copied in the 9th century during the Byzantine Renaissance into three books, Codices A, B, and C. Though Codex A and Codex B were translated and spread into the West (with the originals disappearing), Codex C became a victim of "repurposing" in the 13th century when it was dismantled, scraped, overwritten, and rebound as a religious tract. It contained the only extant description of Archimedes' Method. It eventually made its way into a Greek Orthodox library in Istanbul, where it was found 700 years later by Johann Heiberg in 1906 and published in 1915. But there was more. When the book was dismantled later, writing was found along the spine, where modern imaging techniques in the first decade of the 21st century revealed the meaning that had been hidden for almost a thousand years ([6], [10]).

So to sketch how these ancient treasures made their precarious way from the past to the present I will just highlight the amazingly fortuitous arrival of the Abbasid Caliphate and the founding of Baghdad in 762 AD, some hundred years after the establishment of Islam. Their thirst for knowledge led to the acquisition of documents from the Greeks, Persians, and Indians and to the great Translation Movement to render them into Arabic (Figure 2). One of the difficulties the Arabs faced was where to obtain the Greek documents. The Roman, Latin West had forgotten most of their Greek. The Byzantine East preserved the Greek language, but its Orthodox Christianity no longer cherished the ancient "pagan" knowledge. Details of how the Arabs succeeded are captured in the books of Gutas, Moller, and Wells ([12], [13], [14]). Alas, eventually more conservative Arab

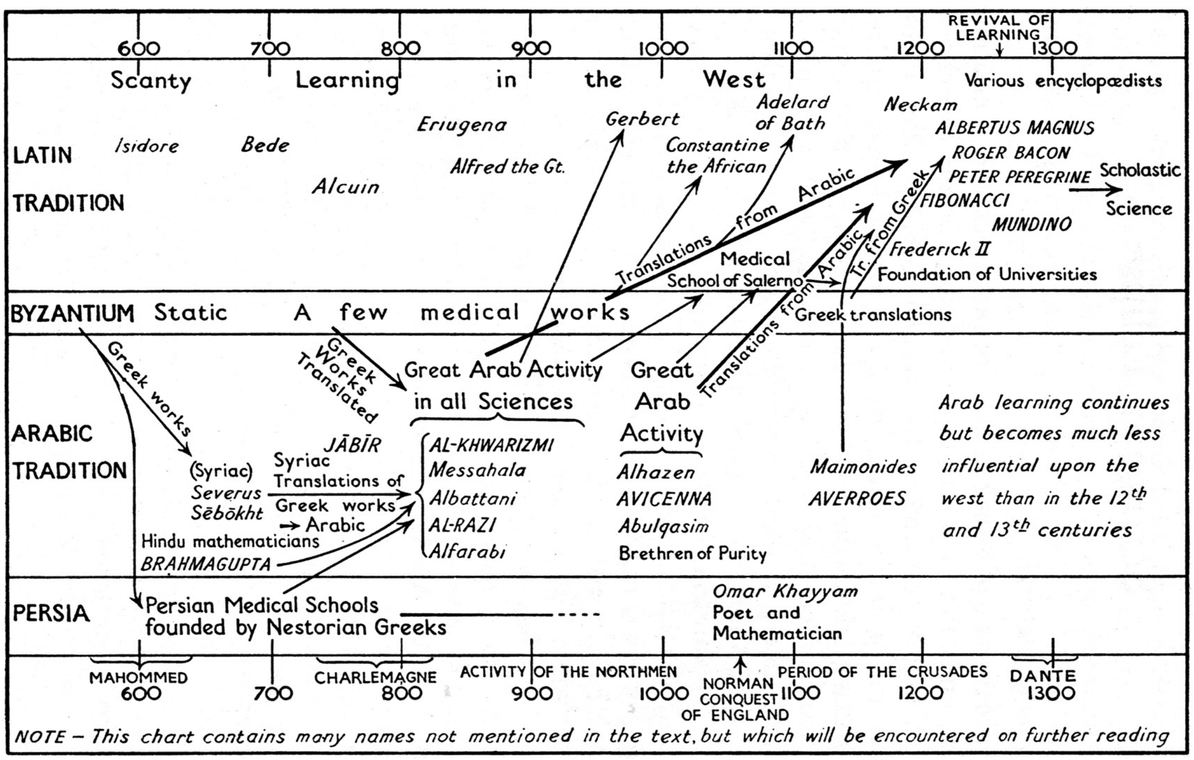


Figure 2 Transfer of Greek Knowledge Timeline ([11] p.51)

factions echoed their earlier like-minded Christians in trying to suppress the flow of ancient wisdom.

A good summary of this "by the skin of our teeth" transmission of Greek thought is from Luciano Canfora's *The Vanished Library* about the library of Alexandria ([16] pp.193-197):

> In the Hellenistic-Roman world, there had been many libraries, but they had been ephemeral. The small city and regional libraries, as well as the great centres, had been emblems—like the hot baths and the gymnasia—of a proud *civilitas* now engulfed in the anarchy of war. …
>
> By the middle of the fourth century, even Rome was virtually devoid of books. …
>
> Surveying this series of foundations, refoundations and disasters, we follow a thread that links together the various, and mostly vain, efforts of the Hellenistic-Roman world to preserve its books. Alexandria is the starting point and the prototype; its fate marks the advent of catastrophe, and is echoed in Pergamum, Antioch, Rome, Athens. …
>
> The great concentrations of books, usually found in the centres of power, were the main victims of these destructive outbreaks, ruinous attacks, sackings and fires. The libraries of Byzantium proved no exception to the rule. In consequence, what has come down to us is derived not from the great centres but from 'marginal' locations, such as convents, and from scattered private copies.

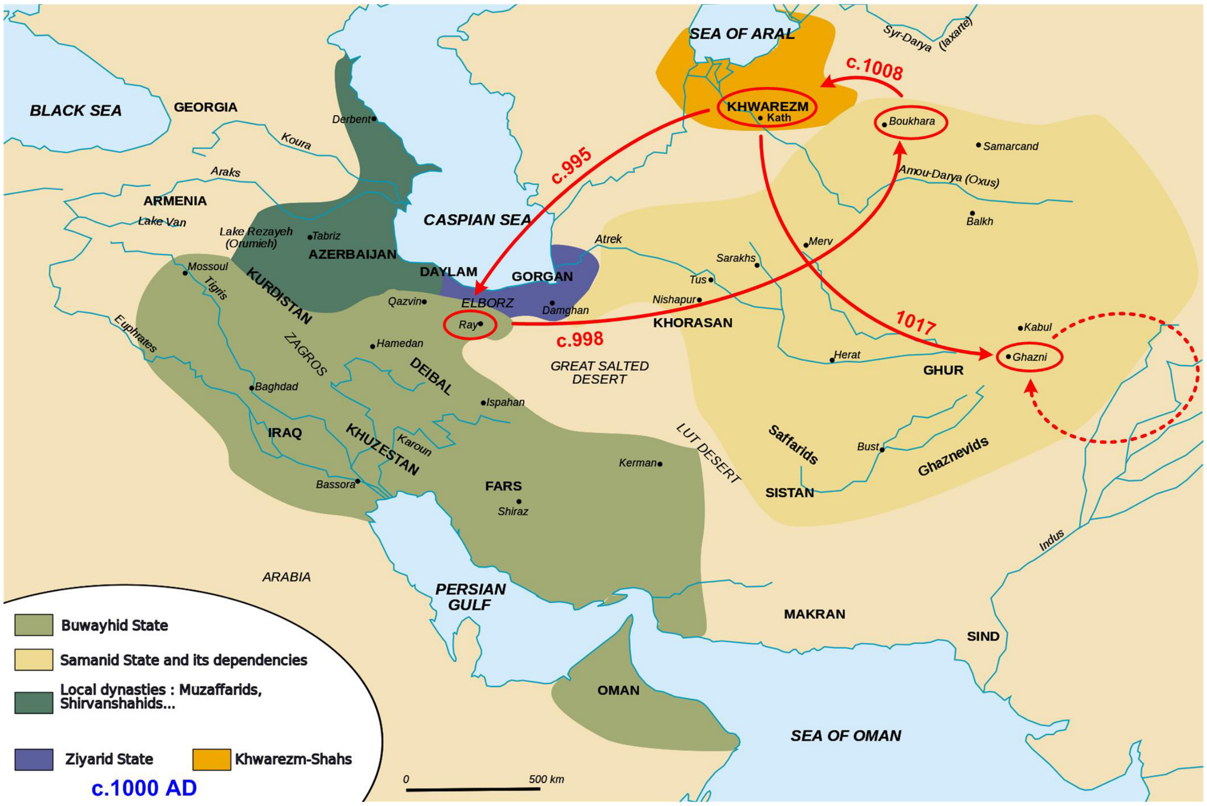
Dominance of Euclidean Geometry

Another feature highlighted in my timeline is the dominance of Euclidean geometry. Besides the content, this manifested itself primarily in the power of its logical method of deductive reasoning. Even though Archimedes had developed a Method that involved lever arms to arrive at the areas and volumes of curved figures and solids, he still resorted to plane geometry to prove his results. For the same reason Newton couched the *Principia* in geometric proofs, even though he had obtained many of his results via the nascent calculus. This geometric dominance was finally eclipsed by the power of symbolic algebra that grew out of the Arab contributions and subsequent paper calculations, and to the growing development of the calculus. Even so, the likes of Jefferson and Lincoln still revered Euclidean geometry as the touchstone of rationalism.

Travel and Communication

Something that is implicit in the timeline above is the peripatetic nature of the individual contributing mathematicians. I admit I sort of viewed all these ancient countries as more or less isolated from one another. But the more I became aware of details I realized there was a great deal of travel and interaction. Greek mathematicians were spread all around the Mediterranean rim in various Greek colonies. There is ample evidence that religious and philosophical thinkers produced letters to one another for information and to resolve disputes. There were of course the great Christian Councils that drew attendees from widespread regions. I was amazed when I first hear about Ashoka sending Buddhist missionaries from India to the Mediterranean around 250 BC. And India has a long tradition of early Christian settlements on its west coast where Jewish communities had been previously established during Roman times to support an active trade throughout the Arabian Sea.

A particular example caught my attention involving the mathematician and philosopher Al-Biruni (973 – after 1050) (Figure 3). As shown, he traveled all over the then extent of Iran and Northern India. But what really caught my attention was that he was from Khwarizm, the same region Al-Khwarismi came from some 200 years before. I had thought of Al-Khwarismi in Baghdad but hadn't realized he originated from a distant land near the disappearing Aral Sea in present-day Uzbekistan. Given modern-day highways, railroads, and air transport, it is hard to imagine the difficulties in travel for these ancient scholars. Accounts of such travels have always been popular and valuable, as exemplified by the tales of the 13th century Marco Polo.



commons.wikimedia.org

Figure 3 Sojourn of Al-Biruni

Besides raising interest in the actual lives and personal achievements of these persons, any extensive travel also feeds the quandary over mathematical discoveries—how many were from isolated individual endeavors versus how many were the result of outside influence. Regarding this debate in Indian mathematics, Plofker ([15]) provides an exhaustive analysis of the pros and cons of Greek influence. She largely supports the independence of the Indian decimal place system, zero as a number, and negative numbers, with the possible influence of the Chinese on the latter two. She does believe Indian astronomy was influenced by the Greeks, especially Ptolemy.

Disclaimer

As I have said numerous times, I am not an expert in the subject matter I have been discussing here. It is sort of a distillation of unfamiliar and surprising aspects of history that have fascinated me down through the years and that I have explored to some extent. I offer the topics in hopes that they might stimulate a similar curiosity on the part of the reader to pursue the inquiries more deeply.

References

[1] Mazur, Joseph, *Enlightening Symbols: A Short History of Mathematical Notation and Its Hidden Powers*, Princeton University Press, Princeton and Oxford, 2014.

[2] Heeffer, Albrecht, “On the Nature and Origin of Algebraic Symbolism,” In Bart Van Kerkhove (ed.), *New Perspectives on Mathematical Practices: Essays in Philosophy and History of Mathematics*. World Scientific. pp. 1-27 (2009) (logica.ugent.be/albrecht/thesis/PMP2007Heeffer.pdf)

[3] Kline, Morris, *Mathematical Thought from Ancient to Modern Times*, Oxford University Press, New York, 1972

[4] Kosmin, Paul J., “A Revolution In Time”, *Aeon*, 7 May 2019. (https://aeon.co/essays/when-time-became-regular-and-universal-it-changed-history, retrieved 5/12/2019)

[5] Jenkins, Philip, “The Missing Century”, *Anxious Bench, Patheos*, 17 November 2014. (http://www.patheos.com/blogs/anxiousbench/2014/11/the-missing-century/, retrieved 12/13/15) Also see Jenkins’s book, *Crucible of Faith: The Ancient Revolution That Made Our Modern Religious World*, Basic Books, New York, 2017, in which he explores in detail the intertestimental period and its implications.

[6] Netz, Reviel and William Noel, *The Archimedes Codex: How a Medieval Prayer Book Is Revealing the True Genius of Antiquity's Greatest Scientist*, Da Capo Press, 2007

[7] Bauer, Susan Wise, *The History of the Renaissance World, From the Rediscovery of Aristotle to the Conquest of Constantinople*, W. W. Norton and Co., New York & London, 2013

[8] Christie, Thony, “Oh, FFS!”, *The Renaissance Mathematicus*, 27 February 2014. (https://thonyc.wordpress.com/2014/02/27/oh-ffs/)

[9] Christie, Thony, “Illuminating the Middle Ages”, *The Renaissance Mathematicus*, 27 April 2022. (https://thonyc.wordpress.com/2022/04/27/illuminating-the-middle-ages/)

[10] “Archimedes Palimpsest”, *Wikipedia* (https://en.wikipedia.org/wiki/Archimedes_Palimpsest, retrieved 1/1/2021)

[11] Sherwood, Taylor, F., *A Short History of Science and Scientific Thought, with Readings from the Great Scientist from the Babylonians to Einstein*, W. W. Norton & Co., New York, 1949.

[12] Gutas, Dimitri, *Greek Thought, Arabic Culture: The Graeco-Arabic Translation Movement in Baghdad and Early 'Abbasid Society (2nd-4th/8th-10th centuries)*, Routledge, New York, 1998.

[13] Moller, Violet, *The Map of Knowledge: A Thousand-Year History of How Classical Ideas Were Lost and Found*, Doubleday, New York, 2019.

[14] Wells, Colin, *Sailing from Byzantium: How a Lost Empire Shaped the World*, Delacorte, 2006, Delta paperback, 2007

[15] P1ofker, Kim, *Mathematics In India*, Princeton University Press, 394 pp, Jan 2009

[16] Canfora, Luciano, *The Vanished Library*, Sellerio editore 1987, Martin Ryle, tr., Hutchinson Radius 1989, Corrected Translation, U. of California Press, 1990.