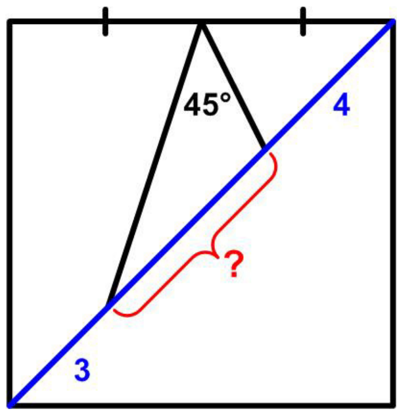
Missing Interval Puzzle

6 July 2022

Jim Stevenson



Henk Reuling posted a deceptively simple-looking geometric problem[1] on Twitter.

I found this old one cleaning up my 'downloads' [source unknown] I haven't been able to solve it, so help!

According to the given information in the figure, what is the length of the missing interval on the diagonal of the square?

My Solution

I relied heavily on Visio. First, I attempted to construct a diagram satisfying the constraints of the problem. In order to get the 45° angle accurately, I constructed an isosceles right triangle and pivoted it about its vertex on the midpoint of the top edge of the square until one leg intersected the diagonal of the square 4 units from the upper right hand corner (Figure 1). Through trial and error with different sized squares, I managed to find one in which the hypotenuse of the right triangle intersected the diagonal 3 units from the lower left corner of the square.

I then happened to notice that if I extended the leg of the right triangle, it seemed to intersect the lower right corner of the square. So I made that an *assumption* for the problem to see where it would lead.

I then set up a coordinate system for the diagram by choosing the origin at the upper right corner of the square. The diagonal became the equation

$$y = x,$$

and the equation for the leg of the isosceles right triangle intersecting the diagonal at the point $(-4/\sqrt{2}, -4/\sqrt{2})$, 4 units from the corner, with slope -2 became the equation

$$y = -2(x + 6\sqrt{2}).$$

This meant the leg of the right triangle intersected the x-axis at $(-6/\sqrt{2}, 0)$ and the y-axis at $(0, -12/\sqrt{2})$. And this meant the diagonal of the square was 12 units long, which implied the length of the missing interval was $12 - 7 = 5$.

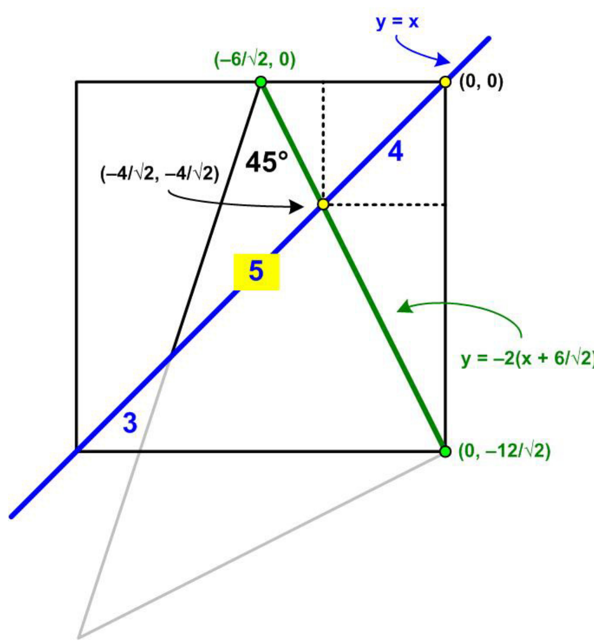


Figure 1

All of this would be true, *if* the hypotenuse of the right triangle with right-angle vertex at the square's lower right corner really intersected the diagonal of the square 3 units from the lower left corner. So I set out to prove that.

[1] 12:50 PM, Jan 5, 2022, (https://twitter.com/HenkReuling/status/1478785993956339716 7/06/2022)

Resorting to Visio again, I noticed that the perpendicular bisector of the leg of the right triangle *appeared* to intersect the diagonal of the square at the point 3 units from the lower left corner (Figure 2). I sought to prove that with the easily derived equation for that perpendicular line. Its slope is the negative reciprocal of the leg of the triangle, or ½. A point on the line would be the midpoint of the leg, or $(–3/\sqrt{2}, –6/\sqrt{2})$. So the equation becomes

$$y = (x - 9/\sqrt{2})/2.$$

Then the intersection with $y = x$ is at the point $(–9/\sqrt{2}, –9/\sqrt{2})$, which is what we want (it is 3 units from the lower left corner).

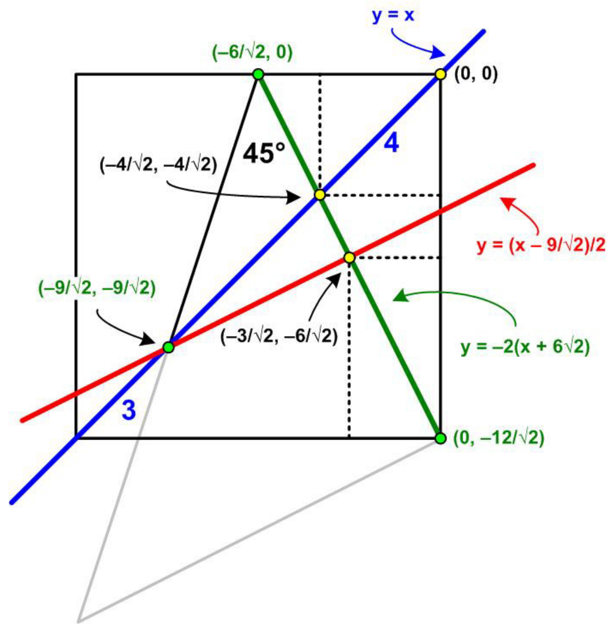


Figure 2

However, to show that the *hypotenuse* of the right triangle also intersects there, we need to show the distances between the midpoint of the leg of the right triangle to each of the vertices are the same. The square of the distance along the green line is

$$(3/\sqrt{2})^2 + (–6/\sqrt{2})^2$$

and the square of the distance along the red line is

$$(6/\sqrt{2})^2 + (3/\sqrt{2})^2$$

and so the same.

Therefore all the assumptions for the problem turned out to be true with the result that the missing interval is 5 units long. (The serendipitous discovery of the isosceles right triangle made this an elegant and delightful problem.)

Reuling Solution

Actually, the solution presented at Reuling's Twitter post[2] was from a commenter "nobodyreally" (https://twitter.com/gamefeast).

nobodyreally @gamefeast Jan 5 Replying to @HenkReuling

I was able to get FG in terms of x (FG = 2*x*sqrt(2) - 7), but after that it turns into a quartic which doesn't have a closed form, so I suspect there's some other trick I'm missing.

Sorry, i'm calling x = |DE| = |CE|

Ok I solved it, answer is 5 (which I actually guessed earlier). It's going to take a while to write this up though, will post later :D

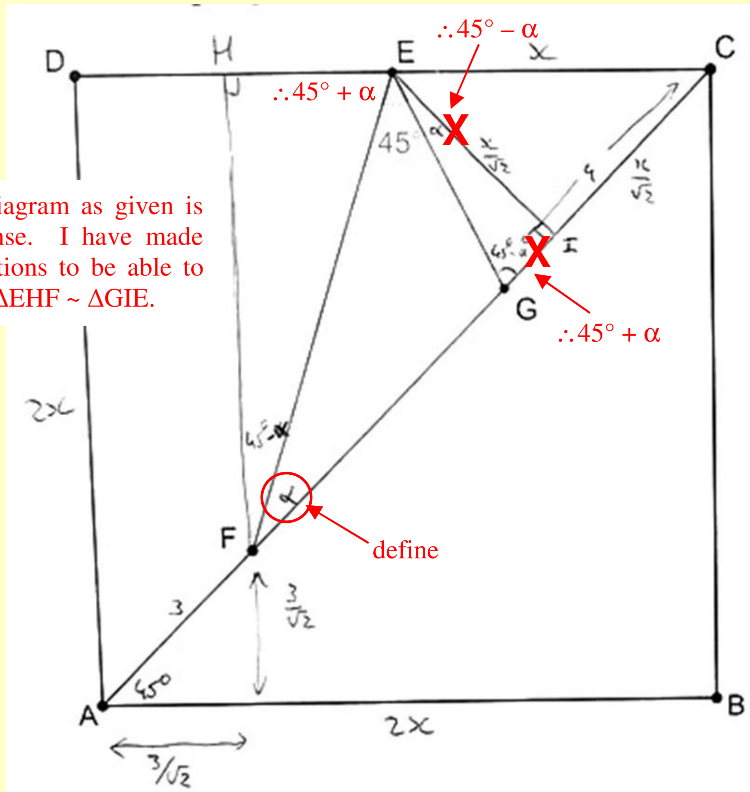
Henk Reuling @HenkReuling· Jan 7

Thank you!! The crucial idea are the similar triangles you start with. After that it's pretty easy. I didn't need Wolfram Alpha to solve the equation ;-)

[2] https://twitter.com/gamefeast/status/1478890440732516353

nobodyreally @gamefeast· Jan 7

It took me a long time to find that. At one point I just labeled alpha on a whim and then started filling out as many other angles as I could in terms of alpha and noticed it. I had so many failed attempts using trig identities



The diagram as given is nonsense. I have made corrections to be able to show ΔEHF ~ ΔGIE.

$EC = x = DF$

Norm $\angle EFG = \alpha$

$\rightarrow \angle EFH = 45° - \alpha$

$\rightarrow \angle \cancel{EGI} = 45° - \alpha$

∠GEI

$\rightarrow \Delta EHF \sim \Delta GIE$

$CI = \frac{x}{\sqrt{2}} = EI$

$GI = 4 - \frac{x}{\sqrt{2}}$

$FH = 2x - \frac{3}{\sqrt{2}}$

$EH = x - \frac{3}{\sqrt{2}}$

$$\frac{IG}{EI} = \frac{EH}{HF} \rightarrow \frac{4 - \frac{x}{\sqrt{2}}}{\frac{x}{\sqrt{2}}} = \frac{x - \frac{3}{\sqrt{2}}}{2x - \frac{3}{\sqrt{2}}} \rightarrow \frac{4\sqrt{2} - x}{x} = \frac{x\sqrt{2} - 3}{2\sqrt{2}x - 3}$$

$$\rightarrow x(x\sqrt{2} - 3) = (4\sqrt{2} - x)(2\sqrt{2}x - 3)$$

$$\sqrt{2}.x^2 - 3x = 16x - 12\sqrt{2} - 2\sqrt{2}x^2 + 3x$$

$$(\sqrt{2} + 2\sqrt{2})x^2 - 22x + 12\sqrt{2} = 0$$

$$3x^2 - 11\sqrt{2}x + 12 = 0$$

$$x = \frac{11\sqrt{2} \pm \sqrt{98}}{6} = \frac{11\sqrt{2} \pm 7\sqrt{2}}{6}$$

$x = \frac{18\sqrt{2}}{6} \vee x = \frac{4\sqrt{2}}{6}$

$x = 3\sqrt{2} \vee x = \frac{2}{3}\sqrt{2}$

clearly $x > \frac{1}{2} \cdot \frac{7}{\sqrt{2}}$

$\rightarrow \boxed{x = 3\sqrt{2}} \rightarrow AB = 6\sqrt{2}$

$\rightarrow AC = 12 \rightarrow FG = 12 - 7 = \boxed{5}$

Comment. I couldn't follow the argument at first, and then I realized the diagram was mislabeled, probably when the solution was written up. I finally figured out minimal changes that would still show ΔEHF similar to ΔGIE. I did not check the rest of the arithmetic.

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