The Maths of Lviv

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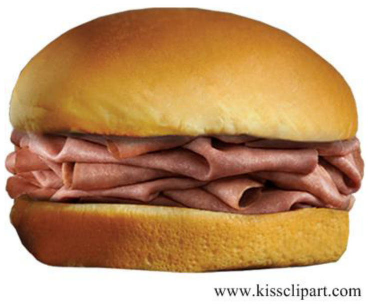


Unfortunately Ukraine has receded from our attention under the threat from our own anti-democratic forces, but this Monday Puzzle[1] from Alex Bellos in March is a timely reminder of the mathematical significance of that country.

Like many of you I've hardly been able to think about anything else these past ten days apart from the war in Ukraine. So today's puzzles are a celebration of Lviv, Ukraine's western city, which played an important role in the history of 20th century mathematics. During the 1930s, a remarkable group of scholars came up with new ideas, methods and theorems that helped shape the subject for decades.

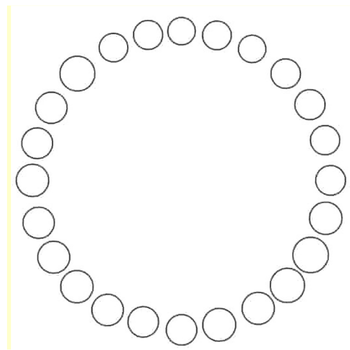
The Lwów school of mathematics – at that time, the city was in Poland – was a closely-knit circle of Polish mathematicians, including Stefan Banach, Stanisław Ulam and Hugo Steinhaus, who made important contributions to areas including set-theory, topology and analysis. …

Of the many ideas introduced by the Lwów school, one of the best known is the "ham sandwich theorem," posed by Steinhaus and solved by Banach using a result of Ulam's. It states that it is possible to slice a ham sandwich in two with a single slice that cuts each slice of bread and the ham into two equal sizes, whatever the size and positions of the bread and the ham.

Today's puzzles are also about dividing food. The first is from Hugo Steinhaus' *One Hundred Problems in Elementary Mathematics*, published in 1938. The second uses a method involved in the proof of the ham sandwich theorem.

1) *Three friends each contribute £4 to buy a £12 ham. The first friend divides it into three parts, asserting the weights are equal. The second friend, distrustful of the first, reweighs the pieces and judges them to be worth £3, £4 and £5. The third, distrustful of them both, weighs the ham on their own scales, getting another result. If each friend insists that their weighings are correct, how can they share the pieces (without cutting them anew) in such a way that each of them would have to admit they got at least £4 of ham?*

2) *Ten plain and 14 seeded rolls are randomly arranged in a circle, equidistantly spaced, as below. Show that using a straight line it is possible to divide the circle into two halves such that there are an equal number of plain and seeded rolls on either side of the line.*



Show there is always a diameter that cuts the circle into two batches of 12 rolls with an equal number of plain and seeded.

Question 2 is adapted from Mathematical Puzzles by Peter Winkler, who gives as a reference Alon and West, The Borsuk-Ulam Theorem and bisection of necklaces, Proceedings of the American Mathematical Society 98 (1986).

[1] 7 March 2022, https://www.theguardian.com/science/2022/mar/07/can-you-solve-it-the-maths-of-lviv

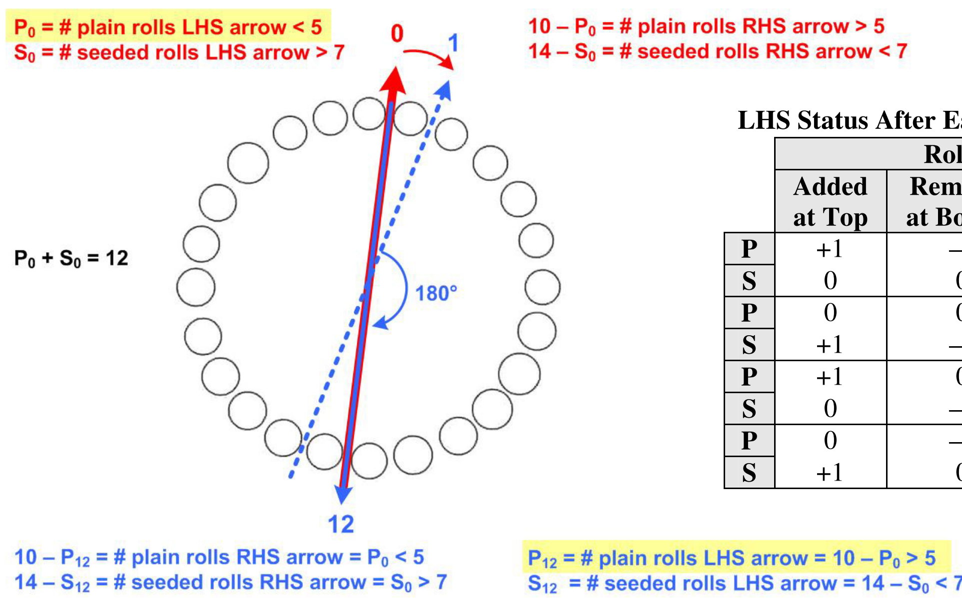
My Solution

Problem 1. Table 1 summarizes the values associated with each friend and the ham portions. Clearly Friend 1 doesn't care what portion they get. So it is up to Friends 2 and 3. Friend 3 can choose any portion they want that they think costs £4 or more (at least one portion must be £4 or more for the estimates to add up to £12). Then Friend 2 can choose either portion 2 or 3, depending on which is not chosen by Friend 3, and Friend 1 can take the remaining portion. Thus all three friends have portions they think cost £4 or more.

Table 1

Cost of Portions		Friend		
		1	**2**	**3**
Ham Portion	**1**	£4	£3	x
	2	£4	£4	y
	3	£4	£5	z

Problem 2. Figure 1 summarizes the solution. We draw an initial (red) diameter arrow through the circle of rolls. Call this position 0. We rotate the arrow one roll clockwise to position 1. We see that we add a roll at the top and remove a roll at the bottom, so the number of rolls to each side of the arrow remains 12. Let P_0 and S_0 be the numbers of plain rolls and seeded rolls respectively on the left hand side (LHS) of the initial arrow. Then $P_0 + S_0 = 12$. The numbers of plain and seeded rolls on the right hand side (RHS) of the initial arrow are then $10 - P_0$ and $14 - S_0$, respectively.



LHS Status After Each Rotation

	Roll		
	Added at Top	**Removed at Bottom**	**Net**
P	+1	–1	0
S	0	0	0
P	0	0	0
S	+1	–1	0
P	+1	0	+1
S	0	–1	–1
P	0	–1	–1
S	+1	0	+1

Figure 1

After each rotation of the diameter arrow, the table in Figure 1 shows all the cases for the change in number of plain and seeded rolls on the LHS. For example, if a plain roll is added at the top and a plain roll is removed at the bottom, then the net change in plain rolls (and in seeded rolls) is zero. Or if a plain roll is added and a seeded roll is removed, then the plain rolls increase and the seeded rolls decrease on the LHS. In all cases the total number of plain and seeded rolls on the LHS (and RHS) remains 12.

Assume without loss of generality that initially the number of plain rolls P_0 on the LHS is less than 5, and so the number of seeded rolls S_0 is greater than 7. Then after 12 rotations past 12 rolls, the numbers of plain and seeded rolls on the LHS, P_{12} and S_{12}, respectively, are the same as the initial numbers on the RHS, that is, $P_{12} = 10 - P_0$ and $S_{12} = 14 - S_0$. That means now $P_0 > 5$ and $S_0 < 7$. Therefore, there must have been a rotation k at which $P_k = 5$ (and so $S_k = 7$) for the LHS. But that means $10 - P_k = 5$ and $14 - S_k = 7$ on the RHS, and so there are the same number of plain and seeded rolls on both sides of the diameter.

These solutions are essentially the same is those given by Alex Bellos.

Remarks.

As Alex noted,

> For readers wanting to know what this has got to the ham sandwich theorem, both of them involve applications of the intermediate value theorem, which states that if a value is moving continuously from A to B, it will pass every value from A to B.

I discussed the Intermediate Value Theorem in my post on "Existence Proofs"[2] and "Existence Proofs II".[3] For a lengthy journey through the ham sandwich theorem and related ideas, see Chinn, W. G. and N. E. Steenrod, *First Concepts of Topology: The Geometry of Mappings of Segments, Curves, Circles, and Disks*, New Mathematical Library, No.18, Random House, Inc., New York, 1966.
(https://archive.org/download/FirstConceptsOfTopology/First%20Concepts%20of%20Topology.pdf)

[2] https://josmfs.net/2021/09/11/existence-proofs/
[3] https://josmfs.net/2021/11/06/existence-proofs-ii/