15 Degree Triangle Puzzle

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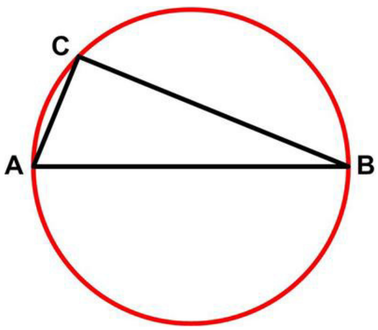


This math problem from Colin Hughes's *Maths Challenge* website (mathschallenge.net) ([1]) is a bit more challenging.

In the diagram, AB represents the diameter, C lies on the circumference of the circle, and you are given that

(Area of Circle) / (Area of Triangle) = 2π.

Prove that the two smaller angles in the triangle are exactly 15° and 75° respectively.

My Solution

Parameterize the diagram as shown in Figure 1. If we can show θ is 30°, then the inscribed angle at B will be half that or 15° and the angle at A will be 90° – 15° = 75°. (Recall that an inscribed triangle with side a diameter must be a right triangle.)

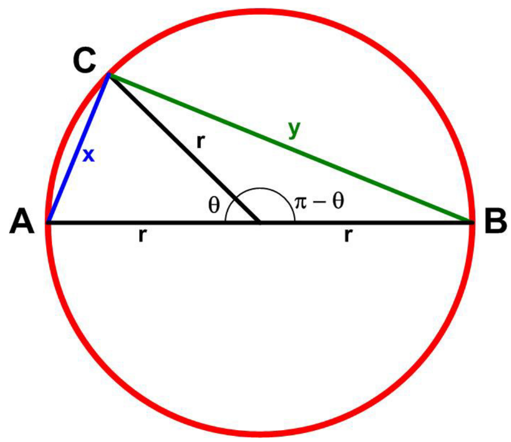


Figure 1

Using the problem constraint on the ratio of areas,

$$\pi r^2 / \tfrac{1}{2} xy = 2\pi,$$

we get

$$r^2 = xy.$$

Now we apply the Law of Cosines[1] twice to get

$$x^2 = 2r^2 - 2r^2 \cos\theta = 2r^2(1 - \cos\theta)$$

$$y^2 = 2r^2 - 2r^2 \cos(\pi - \theta) = 2r^2 + 2r^2 \cos\theta = 2r^2\,(1 + \cos\theta)$$

Multiplying these two equations yields

$$(xy)^2 = 4r^4(1 - \cos^2\theta) \;\Rightarrow\; r^4 = 4r^4(\sin^2\theta) \;\Rightarrow\; \sin^2\theta = \tfrac{1}{4} \;\Rightarrow\; \sin\theta = \tfrac{1}{2} \;\Rightarrow\; \theta = 30°$$

and we are done.

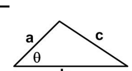
Maths Challenge Solutions

Apparently, there are at least three other ways of solving this problem.

We shall consider three quite different solutions to this problem.

Method 1

As triangle ABC is in a semi-circle, angle ACB is a right angle. Let angle CBA = θ, AB = $2r$, AC = $2r\sin\theta$, and BC = $2r\cos\theta$. Therefore

[1] $c^2 = a^2 + b^2 - 2ab\cos\theta$, where 

$$\text{the area of triangle} = 4r^2 \sin\theta \cos\theta / 2 = 2r^2 \sin\theta \cos\theta,$$

and using the double angle identity: $\sin 2\theta = 2 \sin\theta \cos\theta$,

$$\text{area of triangle} = r^2 \sin 2\theta.$$

So,

$$(\text{Area of Circle}) / (\text{Area of Triangle}) = \pi r^2 / r^2 \sin 2\theta = \pi / \sin 2\theta = 2\pi$$

$$\therefore \sin 2\theta = 1/2 \quad \Rightarrow \quad 2\theta = 30° \quad \Rightarrow \quad \theta = 15°.$$

Hence the complementary angle must be 75 degrees.

Method 2

In the diagram, CD is perpendicular to AB. Let the diameter, AB = $2r$, OC = r, OD = x, and CD = y. Therefore

$$\text{the area of triangle} = 2ry / 2 = ry.$$

So,

$$(\text{Area of Circle})/(\text{Area of Triangle}) =$$

$$\pi r^2 / ry = \pi r / \text{y} = 2\pi \quad \Rightarrow \quad r = 2y$$

Using the Pythagorean Theorem, $x^2 + y^2 = r^2$, so

$$x^2 + y^2 = 4y^2 x^2 = 3y^2 \quad \Rightarrow \quad x = \sqrt{3}\, y.$$

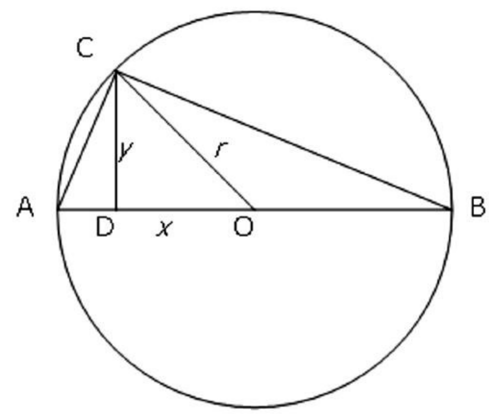


Figure 2

Let angle DOC = θ. Therefore

$$\tan\theta = y/x = y/\sqrt{3}y = 1/\sqrt{3} \quad \Rightarrow \quad \theta = 30°.$$

By using the result that the angle at the centre of a circle (AOC) is twice the angle at the circumference, we deduce that angle ABC = 15°, and it follows that the complementary angle must be 75 degrees.

Method 3

In triangle ABC, let BC = a, AC = b, and AB = c. Because the angle in a semi-circle is a right angle, area of triangle = $ab/2$, and using the Pythagorean theorem, $a^2 + b^2 = c^2$.

As radius, $r = c/2$,

$$r^2 = c^2/4 = (a^2 + b^2)/4.$$

$$\therefore \pi(a^2 + b^2)/4 / (ab/2) = 2\pi$$

$$(a^2 + b^2) / 2ab = 2$$

$$\therefore a^2 + b^2 = 4ab$$

$$\therefore a/b + b/a = 4$$

But in triangle ABC, tan A = a/b and tan B = b/a, so

$$\tan A + \tan B = 4.$$

However,

$$\tan B = \tan(90 - A) = 1/\tan A.$$

Therefore,

$$\tan A + 1/\tan A = 4.$$

By letting $t = \tan A$, we get $t + 1/t = 4$. This leads to the quadratic, $t^2 - 4t + 1 = 0$, which has roots $t = \tan A = 2 \pm \sqrt{3}$.

We will now show that the root corresponding with $\tan A = 2 - \sqrt{3}$ is exactly 15 degrees. Let $T = \tan 15$ and by using the trigonometric identity,

$$\tan 2x = 2\tan x \,/\, (1 - \tan 2x),$$

we get

$$\tan 30 = 1/\sqrt{3} = 2T \,/\, (1 - T^2).$$

This leads to the quadratic,

$$T^2 + 2\sqrt{3}\, T - 1 = 0,$$

which has two roots, $T = -\sqrt{3} \pm 2$. However, as $\tan 15 > 0$, we take the positive root $\Rightarrow$

$$\tan 15 = 2 - \sqrt{3}.$$

Hence the complementary angle must be 75 degrees.

References

[1] Hughes, Colin, "15 Degree Triangle", *Maths Challenge*, (mathschallenge.net) (11 October 2009) #362 p.1. Difficulty: 4 Star. "A four-star problem: A comprehensive knowledge of school mathematics and advanced mathematical tools will be required."