Point of Intersection Problem

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I came across an interesting problem in the *MathsJam Shout* for February 2022 ([1]).

("MathsJam is a monthly opportunity for like-minded self-confessed maths enthusiasts to get together in a pub and share stuff they like. Puzzles, games, problems, or just anything they think is cool or interesting. Monthly MathsJam nights happen in over 70 locations around the world, on the second-to-last Tuesday of each month. To find your nearest MathsJam, visit the website at www.mathsjam.com.")

Given two lines Ax + By + C = 0 and ax + by + c = 0, is there a simple link between the vectors (A, B, C), (a, b, c), and the point where the lines cross?

The answer, of course, is yes, but the question is somewhat open-ended and I was not able to track down any answer given.

Solution

Vectors. If we represent the intersection point (x, y) in the vector form $\mathbf{u} = x\mathbf{i} + y\mathbf{j} + \mathbf{k}$, and the coefficients of the equations as vectors $\mathbf{v} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ and $\mathbf{w} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$, then the two equations

$$Ax + By + C = 0 \text{ and } ax + by + c = 0 \tag{1}$$

become the vector equations

$$\mathbf{u}\cdot\mathbf{v} = \mathbf{0} \text{ and } \mathbf{u}\cdot\mathbf{w} = \mathbf{0}.$$

Therefore, **u** is perpendicular to both **v** and **w**, or equivalently, parallel to $\mathbf{v} \times \mathbf{w}$. This can be expressed vectorially as

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \mathbf{0}. \tag{2}$$

So equation (2) is my answer to the question of a link between the vectors **v** and **w** and the intersection point (x, y).

Algebra. We can make the relationship more explicit by examining the vector components of equation (2). Recall that

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} i & j & k \\ \mathbf{A} & B & C \\ a & b & c \end{vmatrix} = \begin{vmatrix} B & C \\ b & c \end{vmatrix} i - \begin{vmatrix} A & C \\ a & c \end{vmatrix} j + \begin{vmatrix} A & B \\ a & b \end{vmatrix} k$$

Therefore,

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} i & j & k \\ x & y & 1 \\ \begin{vmatrix} B & C \\ b & c \end{vmatrix} & -\begin{vmatrix} A & C \\ a & c \end{vmatrix} & \begin{vmatrix} A & B \\ a & b \end{vmatrix} \end{vmatrix} = \left(\begin{vmatrix} A & B \\ a & b \end{vmatrix} y + \begin{vmatrix} A & C \\ a & c \end{vmatrix}\right) i - \left(\begin{vmatrix} A & B \\ a & b \end{vmatrix} x - \begin{vmatrix} B & C \\ b & c \end{vmatrix}\right) j - \left(\begin{vmatrix} A & C \\ a & c \end{vmatrix} x + \begin{vmatrix} B & C \\ b & c \end{vmatrix} y\right) k = \mathbf{0}$$

implies

$$x = \begin{vmatrix} B & C \\ b & c \end{vmatrix} / D \text{ and } y = -\begin{vmatrix} A & C \\ a & c \end{vmatrix} / D \text{ where } D = \begin{vmatrix} A & B \\ a & b \end{vmatrix}. \tag{3}$$

But this is reminiscent of Cramer's Rule for simultaneous equations. That is, if equations (1) are written

$$\text{Ax + By = –C}$$

$$\text{ax + by = –c,}$$

then Cramer's Rule[1] gives us

$$x = \begin{vmatrix} -C & B \\ -c & b \end{vmatrix} / D \text{ and } y = \begin{vmatrix} A & -C \\ a & -c \end{vmatrix} / D \text{ where } D = \begin{vmatrix} A & B \\ a & b \end{vmatrix}. \tag{4}$$

Applying the rules of determinants to equations (4) that say multiplying a row or column by a constant is equal to multiplying the determinant by that constant, and swapping two rows or two columns negates the determinant, yields equations (3).

Assuming $D \neq 0$ (and therefore $\mathbf{v} \times \mathbf{w} \neq \mathbf{0}$), we get a unique solution for the intersection point (x, y), given the vector equation (2): $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \mathbf{0}$.

Geometry. We can visualize how

$$\text{Ax + By + C = 0}$$

determines a line via the dot product $\mathbf{u} \cdot \mathbf{v} = 0$ from Figure 1. The position vector

$$\mathbf{u} = \text{x}\mathbf{i} + \text{y}\mathbf{j} + \mathbf{k}$$

points to all coordinates of the form (x, y, 1) in the (orange) plane parallel to the xy-plane and one unit above (the z = 1 plane). When **u** is perpendicular to $\mathbf{v} = \text{A}\mathbf{i} + \text{B}\mathbf{j} + \text{C}\mathbf{k}$, it also lies in the (green) plane determined by **v** as its normal. Therefore the points (x, y, 1) determined by **u** lie along the intersection of the orange z = 1 plane and the green plane determined by **v**, namely a line. The projection of this line onto the xy-plane gives us the desired line determined by

$$\text{Ax + By + C = 0.}$$

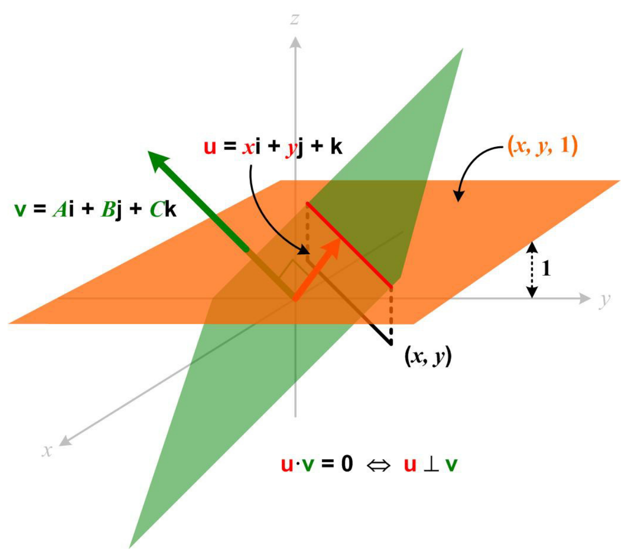

Figure 1

Similarly, ax + by + c = 0 determines a line in the z = 1 plane, and if **v** and **w** are not parallel, the two lines intersect in the desired point (x, y, 1), and thus (x, y).

References

[1] From Colin Beveridge's 2016 MJ Gathering talk, in *MathsJam Shout* February 2022 (https://aperiodical.com/2022/03/mathsjam-leuven-recap-february-2022/). *MathsJam Shout* is a monthly sheet of ideas for activities to do at a MathsJam night.

[1] https://en.wikipedia.org/wiki/Cramer%27s_rule