Seven Girls Puzzle

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This problem comes from the Scottish Mathematical Council (SMC) Senior Mathematical Challenge[1] of 2007 ([1]):

A group of seven girls—Ally, Bev, Chi-chi, Des, Evie, Fi and Grunt—were playing a game in which the counters were beans. Whenever a girl lost a game, from her pile of beans she had to give each of the other girls as many beans as they already had. They had been playing for some time and they all had different numbers of beans. They then had a run of seven games in which each girl lost a game in turn, in the order given above. At the end of this sequence of games, amazingly, they all had the same number of beans—128. How many did each of them have at the start of this sequence of seven games?

My Solution

I began at the end and worked backwards. After game 7, all the girls had $128 = 2^7$ beans. The six other girls had received their previous allotment of beans from Grunt, which meant they doubled their winnings. So these girls went from $64 = 2^6$ beans to $128 = 2^7$ beans. This meant Grunt had to have $2^7 + 6{\cdot}2^6 = 8{\cdot}2^6$ beans at the end of the previous game (game 6) in order to be able to double the other girls' winnings. Therefore after game 6, six of the girls had $2^6 = 64$ beans and Grunt had $8{\cdot}2^6 = 512$ beans. This is the second row in Table 1, the first row being the final result of all girls having 128 beans. The table shows the sequence of events as we progress backwards through the games with all the girls except for the loser of the game halving their winnings each time. The bottom row of the table shows the initial distribution of beans among the girls.

Table 1

	Ally	Bev	Chi-chi	Des	Evie	Fi	Grunt
After game 7	2^7	2^7	2^7	2^7	2^7	2^7	2^7
After game 6	2^6	2^6	2^6	2^6	2^6	2^6	$2^7+6{\cdot}2^6$ $= 8{\cdot}2^6$
After game 5	2^5	2^5	2^5	2^5	2^5	$2^6+5{\cdot}2^5+8{\cdot}2^5$ $= 15{\cdot}2^5$	$8{\cdot}2^5$
After game 4	2^4	2^4	2^4	2^4	$2^5+4{\cdot}2^4+23{\cdot}2^4$ $= 29{\cdot}2^4$	$15{\cdot}2^4$	$8{\cdot}2^4$
After game 3	2^3	2^3	2^3	$2^4+3{\cdot}2^3+52{\cdot}2^3$ $= 57{\cdot}2^3$	$29{\cdot}2^3$	$15{\cdot}2^3$	$8{\cdot}2^3$
After game 2	2^2	2^2	$2^3+2{\cdot}2^2+109{\cdot}2^2$ $= 113{\cdot}2^2$	$57{\cdot}2^2$	$29{\cdot}2^2$	$15{\cdot}2^2$	$8{\cdot}2^2$
After game 1	2	$2^2+2+222{\cdot}2$ $= 225{\cdot}2$	$113{\cdot}2$	$57{\cdot}2$	$29{\cdot}2$	$15{\cdot}2$	$8{\cdot}2$
Initial distribution	$2+447$ $= 449$	225	113	57	29	15	8

[1] http://www.wpr3.co.uk/MC/ [JOS: the Scottish version of the UKMT Challenge]

SMC Solution

The SMC solution ([2]) used the nice idea of framing the losses in terms of the total number of beans T. This made for a more elegant solution than mine.

Notice that the total number of beans they had did not change with each game. So that total was $T = 7 \times 128$. At the end of each game, if the girl who lost had n beans at the start of the game, then she had $2n - T$ beans at the end of that game and all the others had twice as many as when they started that game.

Thus, suppose that the number of beans each girl, in the order given above, had

at the start of the run of seven games was	A	B	C	D	E	F	G
After the next game they had, in order	$2A - T$	$2B$	$2C$	$2D$	$2E$	$2F$	$2G$
After the next game they had, in order	$2^2A - 2T$	$2^2B - T$	2^2C	2^2D	2^2E	2^2F	2^2G
After the next game they had, in order	$2^3A - 2^2T$	$2^3B - 2T$	$2^3C - T$	2^3D	2^3E	2^3F	2^3G
Continuing in this way after seven games they had, in order	$2^7A - 2^6T$	$2^7B - 2^5T$	$2^7C - 2^4T$	$2^7D - 2^3T$	$2^7E - 2^2T$	$2^7F - 2T$	$2^7G - T$

Now $T = 7 \times 128$ and each of these [final] numbers is $128 = 2^7$. So $[2^7G - T =]\ 2^7G - 7 \times 2^7 = 2^7$. So $G = 8$. In the same way $2^7F - 2 \times 7 \times 2^7 = 2^7$ so that $F = 15$. All the others can be deduced in the same way, so that we obtain $E = 29$, $D = 57$, $C = 113$, $B = 225$, $A = 449$.

What the last paragraph in the solution amounts to is in the last line of the table, where each term equals 2^7, divide both sides by 2^7. This yields

A	B	C	D	E	F	G
$64{\cdot}7 + 1 = 449$	$32{\cdot}7 + 1 = 225$	$16{\cdot}7 + 1 = 113$	$8{\cdot}7 + 1 = 57$	$4{\cdot}7 + 1 = 29$	$2{\cdot}7 + 1 = 15$	$7 + 1 = 8$

SMC Alternative Solution

Work backwards in an array:

128	128	128	128	128	128	128
64	64	64	64	64	64	512
32	32	32	32	32	480	256
etc.						

This is the approach I took.

References

[1] "Senior Division: Problems 1" *Mathematical Challenge 2006−2007*, The Scottish Mathematical Council (http://www.wpr3.co.uk/MC-archive/S/S-2006-07-Q1.pdf)

[2] "Senior Division: Problems 1 Solutions" *Mathematical Challenge 2006−2007*, The Scottish Mathematical Council (http://www.wpr3.co.uk/MC-archive/S/S-2006-07-S1.pdf)