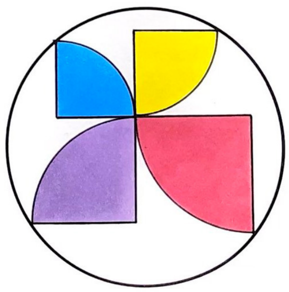
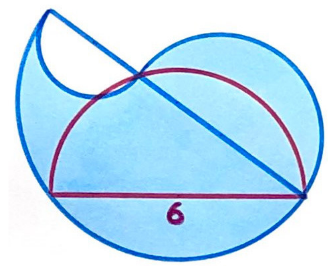
Geometric Puzzle Magic

11 March 2022

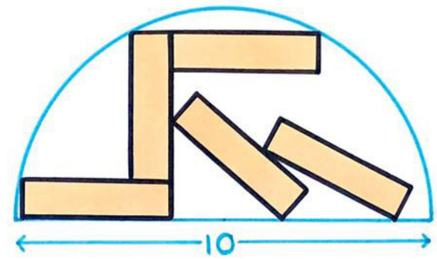
Jim Stevenson



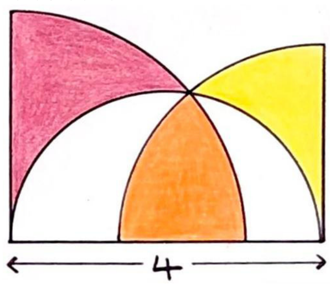
#1 Lopsided Pinwheel. What fraction of the circle do these four quarter circles cover?



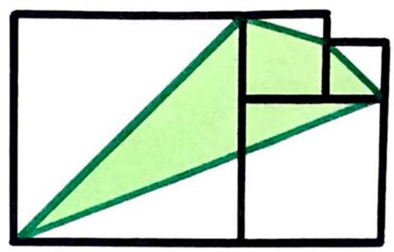
#2 Yin-Yang. The red diameter is 6. What's the total shaded area?



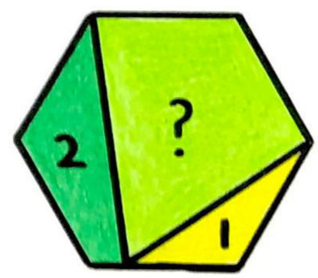
#3 Blocks in Arch. What's the total area of these five congruent rectangles?



#4 Looping Circles. Two quarter circles and a semicircle. What's the total shaded area?



#5 Window View. The green area is 7. What's the total area of the four squares?



#6 Hexagon Tiling. What's the missing area in this regular hexagon?

Here is yet another (belated) collection of beautiful geometric problems from Catriona Agg (née Shearer).

Solution to #1 Lopsided Pinwheel[1]

Since the size of the four quarter circles inscribed in the large circle was arbitrary, we can consider extremes. Figure 1 shows the case where three of the quarter circles have area zero, leaving one quarter circle inscribed in the large circle as shown. Its area is

$$\tfrac{1}{4}\,\pi\,(r\sqrt{2})^2 = \pi\, r^2/2.$$

Thus the ratio of its area to the area of the circle is $\tfrac{1}{2}\pi r^2 / \pi r^2 = \tfrac{1}{2}$.

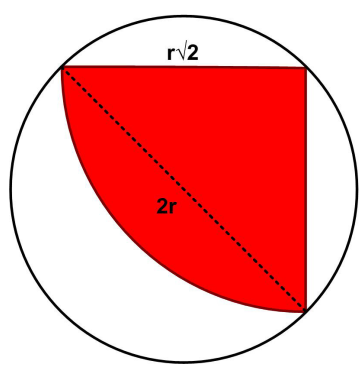


Figure 1

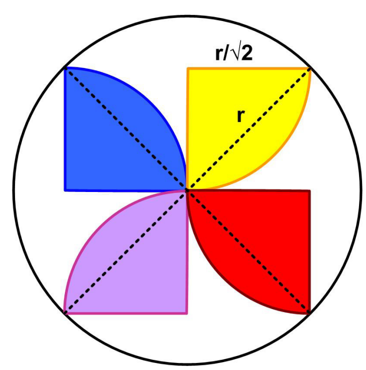


Figure 2

At the other extreme Figure 2 shows the case where all four quarter circles are the same size. Therefore their total area is $4(\tfrac{1}{4})\pi(r\sqrt{2})^2$ =

[1] 8:16 AM • Feb 27, 2021 https://twitter.com/Cshearer41/status/1365651982795628544

$\pi\, r^2/2$. So again the ratio of areas is $\frac{1}{2}\pi r^2 / \pi r^2 = \frac{1}{2}$. We would like to show the ratio of areas is always ½, regardless of the size of the quarter circles.

Figure 3 shows a parameterization of the arbitrary case. It also includes some deductions from the parameters. Since the quarter circles are successive 90° rotations of each other, the chords making four triangles are also rotated 90° and thus perpendicular. We add the central angle θ and its corresponding inscribed angle θ/2. Then we see the pinwheel area is given by

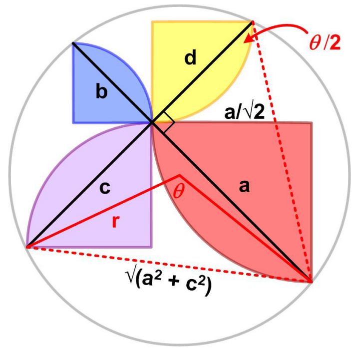


Figure 3

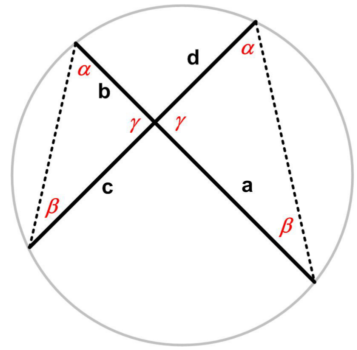


Figure 4 Intersecting Chords Theorem ab = cd

$$A = \tfrac{1}{2}\,(a^2/2 + b^2/2 + c^2/2 + d^2/2)\pi/2 = (a^2 + b^2 + c^2 + d^2)\pi/8 \qquad (1)$$

Figure 4 shows that if we add the dashed lines, we have two similar triangles whose sides satisfy the proportional relation b/c = d/a = t for some constant t, so that b = ct and d = at. Notice that this proportion implies ab = cd.[2] Plugging in the relations b = ct and d = at into equation (1) yields

$$A = (1 + t^2)(a^2 + c^2)\pi/8 \qquad (2)$$

Now from Figure 3

$$\tan \theta/2 = a/d = 1/t$$

so

$$1 + t^2 = 1 + \cos^2 \theta/2 \,/\, \sin^2 \theta/2 = 1/\sin^2 \theta/2$$

From the law of cosines

$$a^2 + c^2 = r^2 + r^2 - 2r^2 \cos \theta = 2r^2(1 - \cos \theta) = 2r^2(2 \sin^2 \theta/2)$$

since $\cos \theta = 1 - 2 \sin^2 \theta/2$. Substituting these results into equation (2) yields

$$A = (1/\sin^2 \theta/2)(4r^2 \sin^2 \theta/2)\pi/8 = 4r^2\pi/8 = \pi\, r^2/2$$

So the ratio of the pinwheel area A to the area of the circumscribed circle C is again A/C = ½.

Comment. When I checked my solution with others on Twitter, I saw a reference to a relation called the "foursquare sum", namely,

$$a^2 + b^2 + c^2 + d^2 = 4\, r^2 \qquad (3)$$

where a, b, c, d are the lengths of the perpendicular chord segments shown in Figure 3 and r is the radius of the circle. Clearly, substituting equation (3) into equation (1) solves the problem immediately.

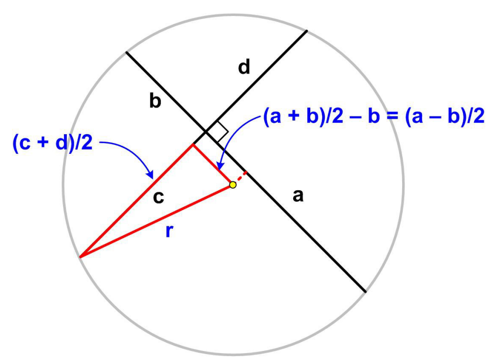


Figure 5

But I wanted to prove the relation. Consider Figure 5. Since the perpendicular bisector of a chord of a circle passes through its center, an application of Pythagoras's Theorem, and the fact that ab = cd, gives us equation (3):

$$r^2 = (c + d)^2/4 + (a - b)^2/4 \;\;\Rightarrow\;\; 4r^2 = c^2 + 2cd + d^2 + a^2 - 2ab + b^2 = a^2 + b^2 + c^2 + d^2$$

[2] JOS: I learned later this is called the Intersecting Chords Theorem. I don't tend to remember many theorems, so I often have to rederive everything as part of solving a problem.

Deriving and using equation (3) certainly solves the problem a lot faster and more directly than using trigonometry as I did. (You still need the Intersecting Chords Theorem result though.)

Solution to #2 Yin-Yang[3]

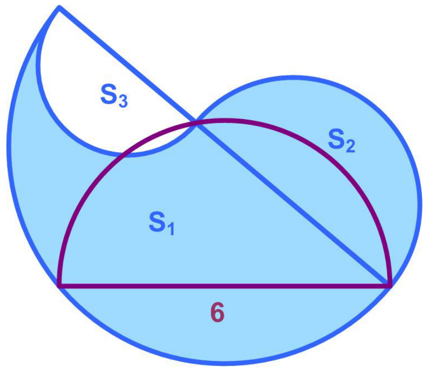


Figure 6

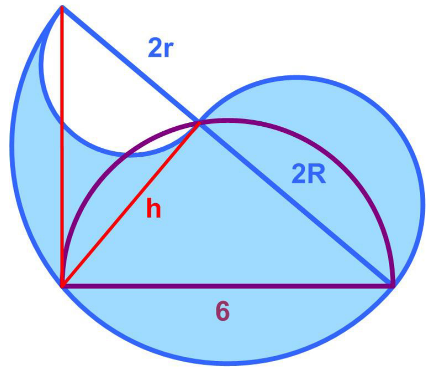


Figure 7

First, label the areas of the non-red semicircles as S_1, S_2, and S_3 in successively diminishing size (Figure 6). Then the desired shaded area is $A = S_1 + S_2 - S_3$. From Figure 7

$$S_1 = \tfrac{1}{2}\pi (R + r)^2, \quad S_2 = \tfrac{1}{2}\pi R^2, \quad S_3 = \tfrac{1}{2}\pi r^2$$

so

$$\text{Area} = \tfrac{1}{2}\pi (2R^2 + 2R\, r)$$

Now from Figure 7 we see that $2R/h = h/2r$, since angles inscribed in semicircles are right angles, and from the Pythagorean Theorem we have that

$$4Rr = h^2 = 36 - 4R^2$$

or

$$R^2 + Rr = 9.$$

So

$$\text{Area} = 9\pi.$$

Solution to #3 Blocks in Arch[4]

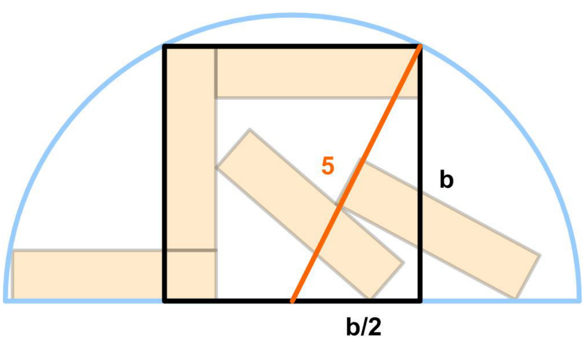


Figure 8

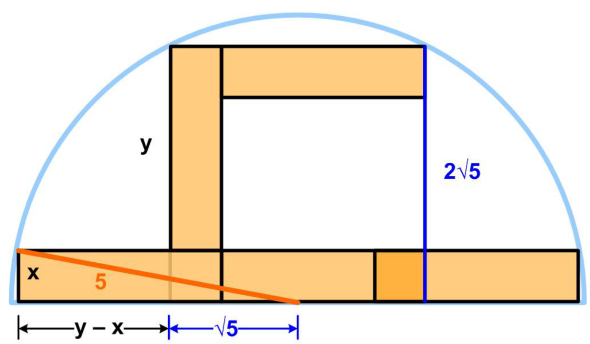


Figure 9

The first thing to notice is that some of the blocks form a square inscribed in the semicircle (Figure 8). By symmetry the midpoint of the bottom edge of the square coincides with the center of the semicircle. Therefore we have a right triangle with hypotenuse (radius of circle) 5 and legs in a 2 : 1 ratio. So

[3] 3:20 AM • Mar 7, 2021 https://twitter.com/Cshearer41/status/1368476681665667073
[4] 4:18 AM • Jul 23, 2021 https://twitter.com/Cshearer41/status/1418485702241697795

$$b^2 + b^2/4 = 25 \quad \Rightarrow \quad b = 2\sqrt{5}.$$

Let x and y be the width and length of the blocks, respectively. From Figure 9 we have

$$x + y = b = 2\sqrt{5}$$

and

$$((y - x) + \sqrt{5})^2 + x^2 = 25$$

or

$$((2\sqrt{5} - 2x)^2 + x^2 = 25$$

or

$$5x^2 - 12\sqrt{5} + 20 = 0.$$

Therefore

$$x = 2/\sqrt{5}$$

$$y = 8/\sqrt{5}$$

So

$$\text{Area of 5 blocks} = 5xy = 16$$

Solution to #4 Looping Circles[5]

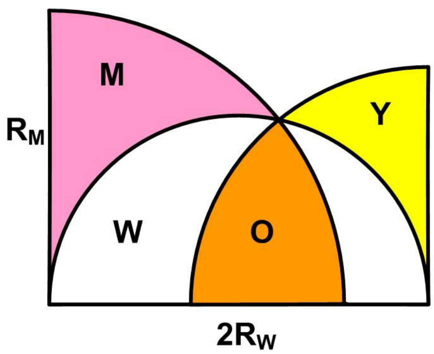


Figure 10

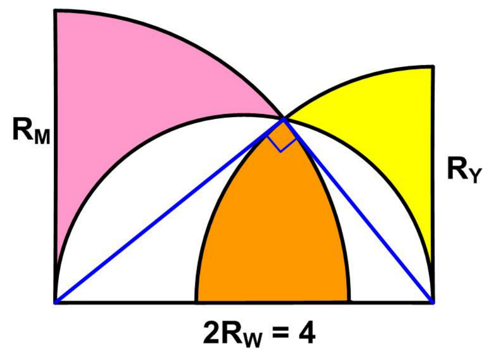


Figure 11

Let M be the area of the magenta quarter circle with radius R_M, Y the area of the yellow quarter circle with radius R_Y, W the area of the white semicircle with radius R_W, and O the area of the orange overlap of the magenta and yellow quarter circles (Figure 10). Then the area of the shaded region is given by

$$\text{Area} = M + Y - W.$$

The intersection of M and Y is O. So O is counted twice in M + Y. Subtracting the semicircle W also removes one of the Os.

Figure 11 shows (blue) radii R_M and R_Y drawn to the point of intersection of all three figures. Since they are drawn from the ends of the diameter of the semicircle, they intersect at a right angle. Therefore,

$$R_M^2 + R_Y^2 = 4^2$$

Thus,

$$\begin{aligned}\text{Area} &= M + Y - W \\ &= \pi R_M^2/4 + \pi R_Y^2/4 - \pi 2^2/2 \\ &= 4\pi - 2\pi = 2\pi\end{aligned}$$

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Solution to #5 Window View[6]

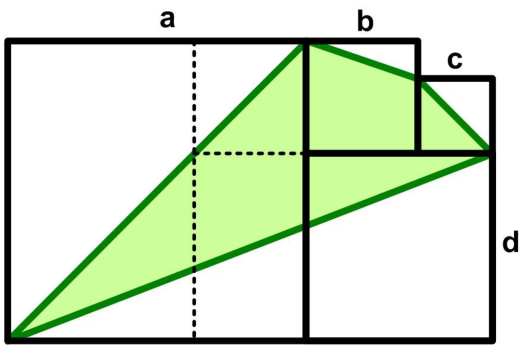

Figure 12

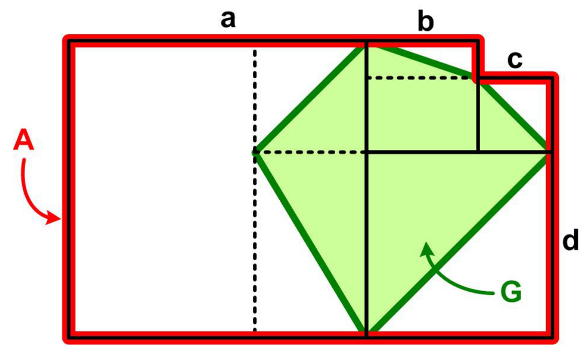

Figure 13

We label the lengths of the squares a, b, c, and d, as shown in Figure 12. Then we have the following relations among the edges: $a = b + d$, $d = b + c \;\Rightarrow\; a = 2b + c$.

Notice that the top edge of the d-square extended intersects the diagonal of the a-square at the lower left corner of a b-square. This line forms the base of triangle in the green region with altitude d. So we can shear the triangle so that the vertex at the lower left corner of the a-square moves to the lower right corner of that square, maintaining the same area (Figure 13).

Now the area of the green region can be computed as

$$G = 7 = \tfrac{1}{2} b^2 + \tfrac{1}{2} c^2 + \tfrac{1}{2} d^2 + \tfrac{1}{2} bd + bc + \tfrac{1}{2} b(b - c)$$

$$14 = 2b^2 + c^2 + d^2 + bd + bc = 2b^2 + c^2 + (b + c)^2 + b(b + c) + bc$$

$$= 2(2b^2 + 2bc + c^2)$$

$$7 = 2b^2 + 2bc + c^2$$

Then the area A of the four squares is given by

$$A = a^2 + b^2 + c^2 + d^2 = (2b + c)^2 + b^2 + c^2 + (b + c)^2 = 3(2b^2 + 2bc + c^2)$$

So $$A = 3 \cdot 7 = 21$$

Solution to #6 Hexagon Tiling[7]

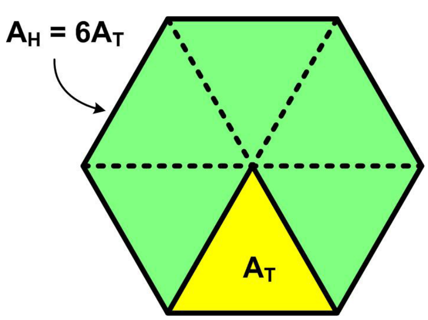

Figure 14

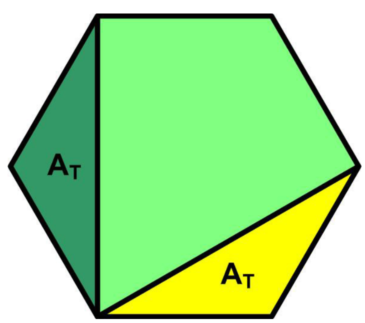

Figure 15

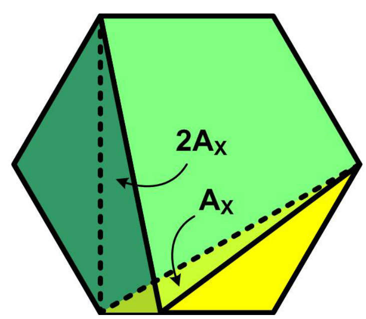

Figure 16

Let A_T be the area of one of the equilateral triangles making up the hexagon (Figure 15). And let A_H be the area of the hexagon. Therefore, $A_H = 6\,A_T$. Shear the yellow equilateral triangle to the position shown in Figure 15. This still has area A_T. The dark green region in Figure 15 also has area

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[6] 5:35 AM • Mar 21, 2021 https://twitter.com/Cshearer41/status/1373568923317194753

[7] 7:12 AM • Mar 10, 2021 https://twitter.com/Cshearer41/status/1369622078417166338

A_T.

Now move the lower left vertex of the yellow triangle along to the right until its area is 1 and the dark green area is 2, as stated in the problem (Figure 16). Call the area removed from the original yellow triangle by this move A_X. Then the added area to the dark green triangle is 2 A_X, since its base is the same and its altitude is twice that of the yellow triangle. Then we have the following relations

$$2 = A_T + 2A_X$$

$$1 = A_T - A_X$$

Doubling the second equation and adding it to the first yields

$$4 = 3A_T.$$

Therefore the unknown area is

$$A_H - 3 = 6A_T - 3 = 6(4/3) - 3 = 5$$