Challenging Triangle Problem

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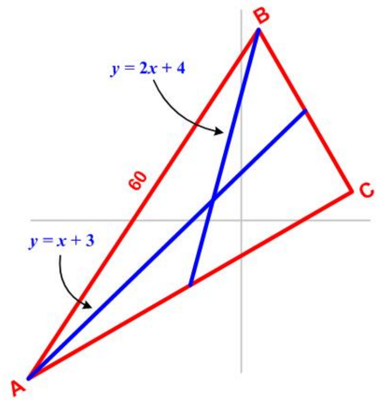


This is a challenging problem from the 1986 American Invitational Mathematics Exam (AIME) ([1]).

Let triangle ABC be a right triangle in the xy-plane with a right angle at C. Given that the length of the hypotenuse AB is 60, and that the medians through A and B lie along the lines $y = x + 3$ and $y = 2x + 4$ respectively, find the area of triangle ABC.

I have included a sketch to indicate that the sides of the right triangle are not parallel to the Cartesian coordinate axes.

The AIME (American Invitational Mathematics Examination) is an intermediate examination between the American Mathematics Competitions AMC 10 or AMC 12 and the USAMO (United States of America Mathematical Olympiad).[1] All students who took the AMC 12 (high school 12th grade) and achieved a score of 100 or more out of a possible 150 or were in the top 5% are invited to take the AIME. All students who took the AMC 10 (high school 10th grade and below) and had a score of 120 or more out of a possible 150, or were in the top 2.5% also qualify for the AIME.[2]

My Solution

I have added the angles α and β and sides a and b to the original figure (Figure 1). The key to the solution (which took me some time to realize) is the slopes of the medians, which can be interpreted as tangents of angles.

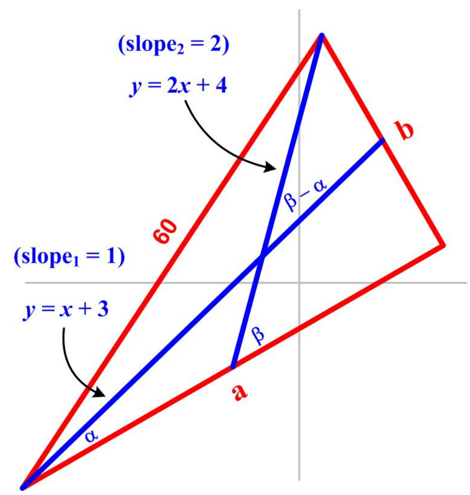


Figure 1

Using the slopes of the medians (blue lines) from the equations, we get

$$\tan(\beta-\alpha)=\frac{slope_2-slope_1}{1+slope_2 slope_1}=\frac{1}{3}$$

But we also have

$$\tan(\beta-\alpha)=\frac{\tan\beta-\tan\alpha}{1+\tan\beta\tan\alpha}$$

Now $\tan\beta = b/(a/2) = 2\ b/a$ and $\tan\alpha = (b/2)/a = (1/2)b/a$. Therefore,

$$\frac{1}{3}=\tan(\beta-\alpha)=\frac{2(b/a)-(1/2)(b/a)}{1+(b/a)^2}=\frac{(3/2)ab}{60^2}$$

or $\quad$ Area ABC = $ab/2 = 60^2/3^2 = 400$

1 https://www.maa.org/math-competitions/about-amc

2 https://www.maa.org/math-competitions/american-invitational-mathematics-examination-aime

AIME Solutions

AIME provides three solutions. The second solution starts out being the same as mine, and then veers into the more complicated path of solving a quadratic equation.

Solution 1

Translate so the medians are $y = x$, and $y = 2x$, then model the points $A : (a, a)$ and $B : (b, 2b)$. (0, 0) is the centroid,[3] and is the average of the vertices, so $C : (-a - b, -a - 2b)$ [Figure 2]. $AB = 60$ so

$$3600 = (a-b)^2 + (2b-a)^2$$
$$3600 = 2a^2 + 5b^2 - 6ab \quad (1)$$

AC and BC are perpendicular, so the product of their slopes is –1, giving

$$\left(\frac{2a+2b}{2a+b}\right)\left(\frac{a+4b}{a+2b}\right) = -1$$
$$2a^2 + 5b^2 = -\frac{15}{2}ab \quad (2)$$

Combining (1) and (2), we get $ab = -800/3$.

Using the determinant product for area of a triangle (this simplifies nicely, add columns 1 and 2, add rows 2 and 3), the area is $|3ab/2|$, so we get the answer to be 400.

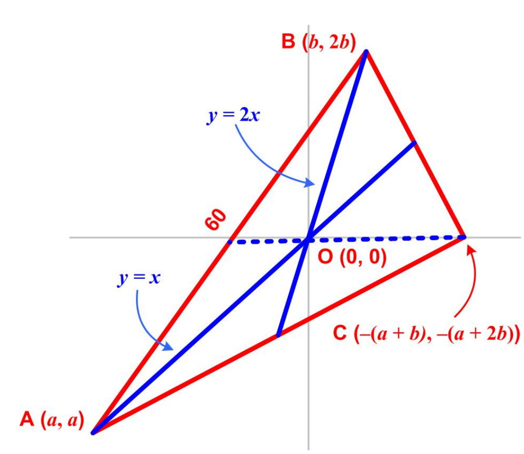


Figure 2 AIME Solution 1

$$\text{Area} = |\,\text{AC} \times \text{AB}\,| / 2 = \begin{vmatrix} i & j & k \\ -2a-b & -2a-2b & 0 \\ b-a & 2b-a & 0 \end{vmatrix} / 2 = \begin{vmatrix} -2a-b & -b \\ b-a & b \end{vmatrix} / 2 = \begin{vmatrix} -3a & 0 \\ b-a & b \end{vmatrix} / 2 = |-3ab/2|$$

Solution 2

The only relevant part about the xy plane here is that the slopes of the medians determine an angle between them that we will use. This solution uses the tangent subtraction identity

$$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}.$$

Therefore, the tangent of the acute angle between the medians from A and B will be

$$\frac{2-1}{1+2\cdot 1} = \frac{1}{3}.$$

Let $AC = x$ and $BC = y$. Let the midpoint of AC be M and the midpoint of BC be N. Let the centroid be G. By exterior angles, $\angle CMB - \angle CAN = \angle AGM$. However we know that since $\angle AGM$ is the acute angle formed by the medians, $\tan \angle AGM = 1/3$. We can express the tangents of the other two angles in terms of x and y.

$$\tan\angle CMB = \frac{y}{\frac{x}{2}} = \frac{2y}{x}$$

[3] The centroid is the point of intersection of the medians of a triangle. The coordinates of the centroid of a coordinatized triangle are (a,b) where a is the arithmetic mean of the x-coordinates of the vertices of the triangle and b is the arithmetic mean of the y-coordinates of the triangle.

while $$\tan\angle CAN = \frac{\frac{y}{2}}{x} = \frac{y}{2x}.$$

For simplification, let $\frac{y}{x} = r$. By the tangent subtraction identity,

$$\frac{2r - \frac{r}{2}}{1 + 2r\cdot\frac{r}{2}} = \frac{3r}{2(1+r^2)} = \frac{1}{3}.$$

We get the quadratic $2r^2 - 9r + 2 = 0$, which solves to give

$$r = \frac{9 \pm \sqrt{65}}{4}.$$

It does not matter which one is picked, because the two roots multiply to 1, so switching from one root to another is like switching the lengths of AC and BC. We choose the positive root and we can plug

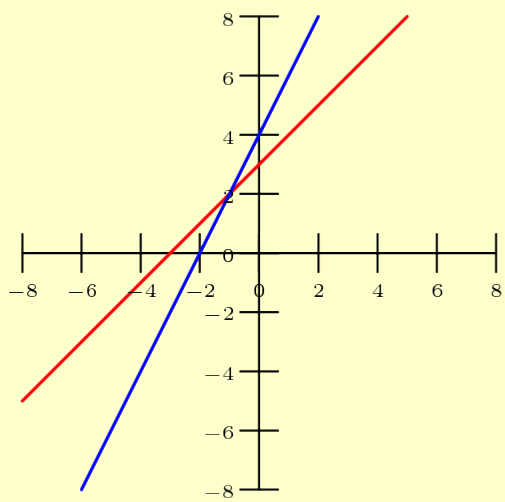
$$y = \left(\frac{9+\sqrt{65}}{4}\right)x$$

into of course the PYTHAGOREAN THEOREM $x^2 + y^2 = 3600$ and solve for $x^2 = 200(9-\sqrt{65})$. We want the area which is

$$\frac{xy}{2} = \frac{x^2(\frac{9+\sqrt{65}}{4})}{2} = \frac{200(9-\sqrt{65})(\frac{9+\sqrt{65}}{4})}{2} = \frac{200\cdot\frac{16}{4}}{2} = 400.$$

Solution 3

We first seek to find the angle between the lines $y = x + 3$ and $y = 2x + 4$.



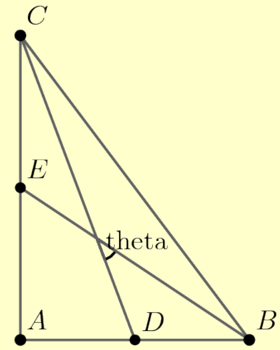
Let the acute angle the red line makes with the x-axis be α and the acute angle the blue line makes with the x-axis be β. Then, we know that $\tan\alpha = 1$ and $\tan\beta = 2$. Note that the acute angle between the red and blue lines is clearly $\beta - \alpha$. Therefore, we have that:

$$\tan(\beta-\alpha) = \frac{2-1}{1+2} = \frac{1}{3}$$

It follows that $\cos(\beta-\alpha) = 1/\sqrt{10}$ and $\sin(\beta-\alpha) = 3/\sqrt{10}$.[4] From now on, refer to $\theta = \beta - \alpha$

[4] JOS: No, $\cos(\beta-\alpha) = 3/\sqrt{10}$ and $\sin(\beta-\alpha) = 1/\sqrt{10}$.

Suppose ABC is our desired triangle. Let E be the midpoint of CA and D be the midpoint of AB such that $CE = EA = m$ and $AD = DB = n$. Let G be the centroid of the triangle (in other words, the intersection of EB and CD). It follows that if $CG = 2x$, $GD = x$ and if $GB= 2y$, $GE = y$.



By the Law of Cosines on ΔGEB, we get that:

$$GE^2 + GC^2 - 2GE \cdot GC \cdot \cos\theta = CE^2 \iff$$
$$(2x)^2 + y^2 - (2x)(y)(2)\cos\theta = m^2$$

By the Law of Cosines on ΔGDB, we get that:

$$DG^2 + GB^2 - 2DG \cdot GB \cdot \cos\theta = DB^2 \iff$$
$$(2y)^2 + x^2 - (2y)(x)(2)\cos\theta = n^2$$

Adding yields that: $$5x^2 + 5y^2 - 8xy\cos\theta = m^2 + n^2$$

However, note that $$(2m)^2 + (2n)^2 = 60^2 \iff m^2 + n^2 = 30^2.$$

Therefore, $$5x^2 + 5y^2 - 8xy\cos\theta = 30^2$$

We also know that by the Law of Cosines on ΔCGB,

$$CG^2 + GB^2 - 2CG \cdot GB \cdot \cos(180 - \theta) = CB^2 \iff$$
$$(2x)^2 + (2y)^2 + 2(2x)(2y)\cos\theta = 60^2 \iff$$
$$x^2 + y^2 + 2xy\cos\theta = 30^2 \iff$$
$$5x^2 + 5y^2 + 10xy\cos\theta = 30^2 \cdot 5.$$

Subtracting this from the $$5x^2 + 5y^2 - 8xy\cos\theta = 30^2$$

we got earlier yields that: $$xy\cos\theta = 200$$

But recall that cos $\theta = 3/\sqrt{10}$ to get that: $$xy = \frac{200\sqrt{10}}{3}$$ [5]

Plugging this into $$x^2 + y^2 + 2xy\cos\theta = 30^2,$$

we get that: $$x^2 + y^2 = 500$$

Aha! How convenient. Recall that

$$(2x)^2 + y^2 - (2x)(y)(2)\cos\theta = m^2$$

and

$$(2y)^2 + x^2 - (2y)(x)(2)\cos\theta = n^2.$$

Then, we clearly have that:

$$m^2n^2 = ((2x)^2 + y^2 - (2x)(y)(2)\cos\theta)((2y)^2 + x^2 - (2y)(x)(2)\cos\theta = n^2) \iff$$
$$m^2n^2 = 17x^2y^2 - 4xy\cos\theta(5x^2 + 5y^2) + 16x^2y^2\cos^2\theta + 4x^4 + 4y^4 \iff$$
$$m^2n^2 = 17 \cdot \frac{200^2 \cdot 10}{9} - 800(5 \cdot 500) + 16 \cdot 200^2 + 4\left((x^2 + y^2)^2 - 2x^2y^2\right) \iff$$
$$\frac{m^2n^2}{100^2} = \frac{680}{9} - 8 \cdot 5 \cdot 5 + 16 \cdot 2^2 + 4\left(5^2 - 2 \cdot \frac{2^2 \cdot 10}{9}\right) \iff$$

[5] JOS: Unnecessary. Can skip this line.

$$\frac{m^2n^2}{100^2} = 4$$

But note that the area of our triangle is $2m \cdot 2n \cdot \frac{1}{2} = 2mn$. As $mn = 200$, we get a final answer of 400.

References

[1] "Problem 15" 1986 AIME Problems (https://artofproblemsolving.com/wiki/index.php/1986_AIME_Problems)