Two Equilateral Triangles

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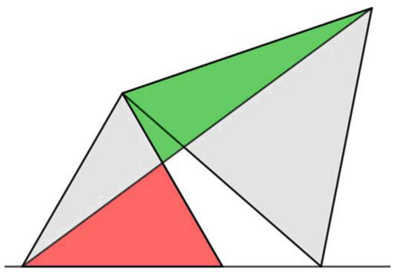
Jim Stevenson



This is a most interesting problem proposed by Mirangu[1] and retweeted by Catriona Agg:

Two equilateral triangles share a vertex. What is the proportion red : green?

My Solution

Since the position of the second, right-hand equilateral triangle was not specified, that meant it was free to pivot about the vertex of the left-hand equilateral triangle, so long as its lower vertex remained along the horizontal line. Starting with the lower vertex coinciding with the lower right vertex of the left-hand triangle, we see that the line connecting the vertices cuts the two figures in half, making the red and green areas equal. But since the moving right-hand triangle is increasing in size, the equality may not hold and there may be a proportion involved. But then the answer would vary with the triangle and not be a constant result, contrary to expectation. So I suspect the answer is the ratio is equal to 1, but that needs proof.

Playing with Visio, I noticed that the upper vertex of the pivoting triangle seemed to follow a straight line at a 60° angle from the lower right vertex of the left-hand triangle (Figure 1). I realized this was another instance of a curve being duplicated by pivoting similar triangles and canted by an angle equal to that at the pivot vertex, in this case 60°.[2]

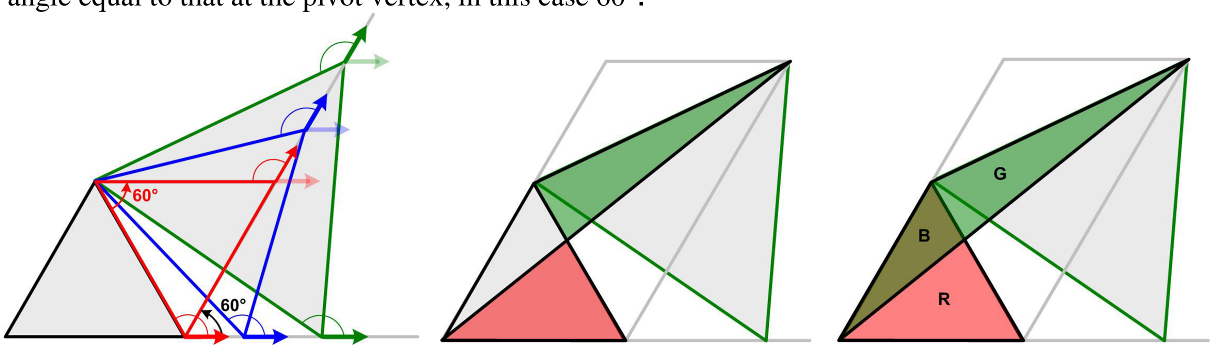


Figure 1 **Figure 2** **Figure 3**

This means the line along which the vertices of the right-hand triangle move is parallel to the left edge of the left-hand triangle (Figure 2). Suppose in Figure 2 the green triangle has area G and the red triangle area R. If we extend these triangles to the left edge of the left-hand equilateral triangle, we see they have a common overlap of area B. Since these extended triangles have a common base (left edge of the left-hand triangle) and common altitude (due to the parallel lines), they have a common area, so that

$$G + B = R + B \;\Rightarrow\; G = R.$$

This means R : G = 1, as predicted.

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[1] https://mirangu.com/down-to-earth/

[2] See for instance "Geometric Puzzle Mindbogglers", in particular, the Swinging Triangle puzzle (which actually is the same situation as here) and the Dancing Rectangle puzzle (http://josmfs.net/wordpress/wp-content/uploads/2021/10/Geometric-Puzzle-Mindbogglers-210927.pdf)

Comment. I had been looking at old problems I posted and suddenly was surprised to realize the result in Figure 1 is exactly the same as in the "Straight and Narrow Problem".[3] There I solved the problem with calculus.

Mirangu Solution

The solution found at Mirangu's Twitter page[4] is attributed to Asmit Dey.[5] I will first provide the solution as is (Figure 4).

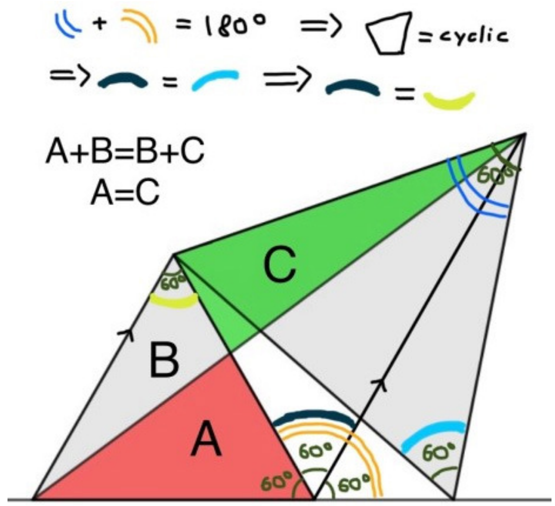

Figure 4 Asmit Dey Solution

It took me a while to unravel what was going on. First, I showed the indicated quadrangle was cyclic, that is, the opposite angles were supplementary[6] (Figure 5): $\alpha + \beta = 180°$, and $\gamma + \delta = 180° - 120° = 60°$ implies $(\gamma + 60°) + (\delta + 60°) = 180°$.

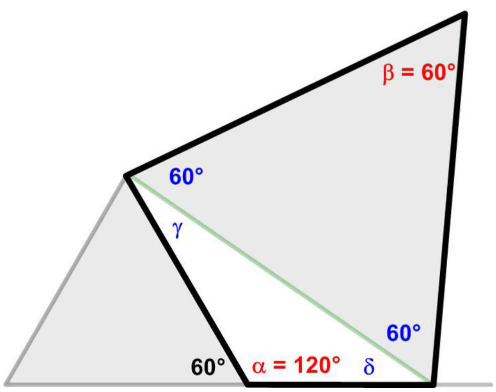

Figure 5

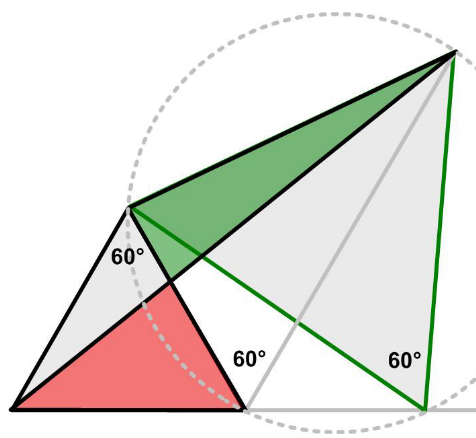

Figure 6

Then the inscribed angles are equal (Figure 6) and their common value is 60°, which implies the grey line is parallel to the left edge of the left-hand triangle. This is the result we showed in Figure 2. The rest of Dey's solution is essentially the same as mine.

[3] http://josmfs.net/2019/03/20/straight-and-narrow-problem/
[4] https://mirangu.com/down-to-earth/
[5] https://twitter.com/Asmit_Dey_
[6] **cyclic polygon** (http://www.mathematicsdictionary.com/english/vmd/full/c/cyclicpolygon.htm). A polygon whose vertices all lie on the same circle. All triangles, all rectangles, and all regular polygons are cyclic. Convex quadrilaterals, whose opposite angles are supplementary, are also cyclic. [JOS: see below p.3]

Comment. Given my lack of familiarity with "cyclic polygons", I had to look them up on the web. There was surprisingly little information. The Wikipedia entry started off with conjectures from as recently as 1995, which did not bode well for firm information. Then I found the entry indicated in footnote 6. So "Convex quadrilaterals, whose opposite angles are supplementary, are also cyclic." Certainly, cyclic quadrilaterals (inscribed in a circle) would have opposite angles summing to 180°, but why was the converse true? No proofs were included, so I created my own.

Claim. *Convex quadrilaterals, whose opposite angles are supplementary, are also cyclic.*

Proof. Pass a circle through three of the vertices of the quadrilateral. Then the fourth vertex either lies inside the circle (Figure 7), or outside the circle (Figure 8), or on the circle. Let θ be the angle at this fourth vertex and ψ the angle at the vertex opposite. Let α be the arc of the circle subtended by the angle θ (Figure 7). We are assuming $\theta + \psi = 180°$. But $\psi = (360° - \alpha)/2$, so this means $\theta = \alpha/2$.

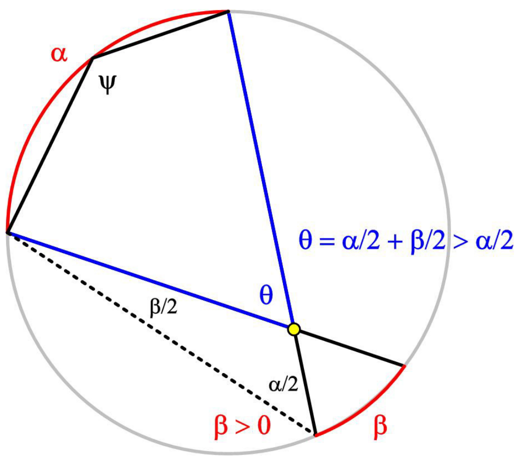

Figure 7 Fourth vertex inside circle of other three.

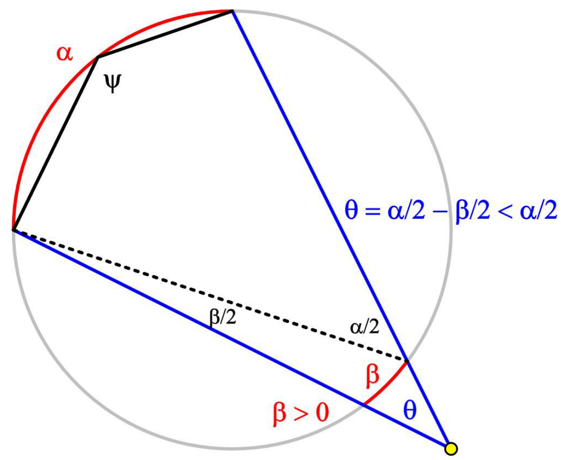

Figure 8 Fourth vertex outside circle of other three.

Case 1. Fourth vertex is inside the circle (Figure 7)

Let β be the arc of the circle subtended by the sides of angle θ extended where they intersect the circle. Since the fourth vertex is not on the circle, $\beta > 0$. From the figure with the added dashed line we see that

$$\theta = \alpha/2 + \beta/2 > \alpha/2,$$

which contradicts that $\theta = \alpha/2$.

Case 2. Fourth vertex is outside the circle (Figure 8)

Again let β be the arc of the circle subtended by the sides of angle θ where they intersect the circle. And again, since the fourth vertex is not on the circle, $\beta > 0$. From the figure with the dashed line we see that

$$\theta = \alpha/2 - \beta/2 < \alpha/2,$$

which again contradicts that $\theta = \alpha/2$.

So the fourth vertex must lie on the circle. This argument can be applied to any opposite angles summing to 180°, thus proving a quadrilateral is cycle if and only if its opposite angles are supplementary.