Playing with Triangles

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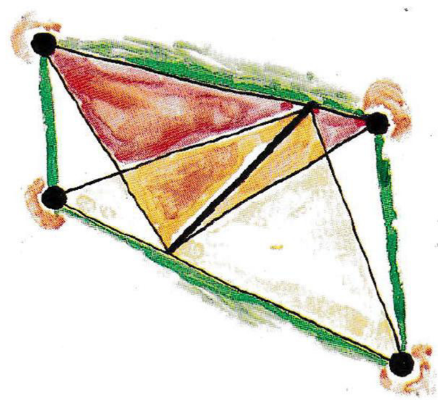
Jim Stevenson



Pavel Chernusky

Here is another elegant *Quantum* math magazine Brainteaser from the imaginative V. Proizvolov ([1]).

Two isosceles right triangles are placed one on the other so that the vertices of each of their right angles lie on the hypotenuse of the other triangle (see the figure at left). Their other four vertices form a quadrilateral. Prove that its area is divided in half by the segment joining the right angles. (V. Proizvolov)

My Solution

I solved the problem purely with plane geometry this time, though with a heavy use of the shearing argument.

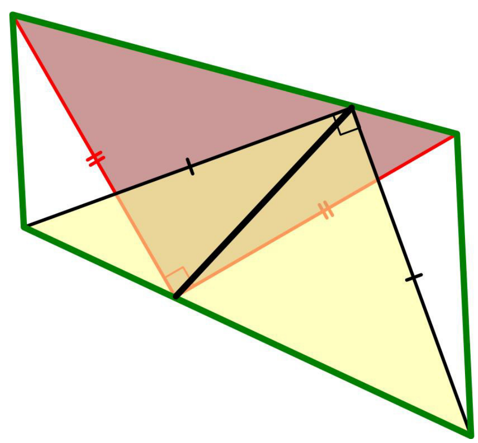


Figure 1 Problem Statement

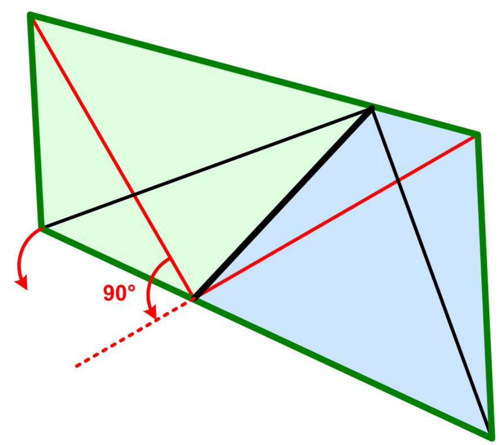


Figure 2 Rotate Left "Half" 90°

Figure 1 shows the isosceles right triangles in the quadrilateral. Rotate the left "half" of the severed quadrilateral 90° (Figure 2). Figure 3 shows the result. Since the red lines were originally the legs of an isosceles right triangle, they now form one straight line of equal line segments. Rotating the left black leg of the other isosceles right triangle 90° has now left it parallel to its partner leg.

Now draw horizontal lines through the vertices of the two "half" quadrilaterals parallel to the red line (Figure 3). The fact that the black legs are parallel and the same length means the intervals between the corresponding parallel lines through their endpoints are also the same distance h. We now employ the shear argument to move the vertices horizontally until the black

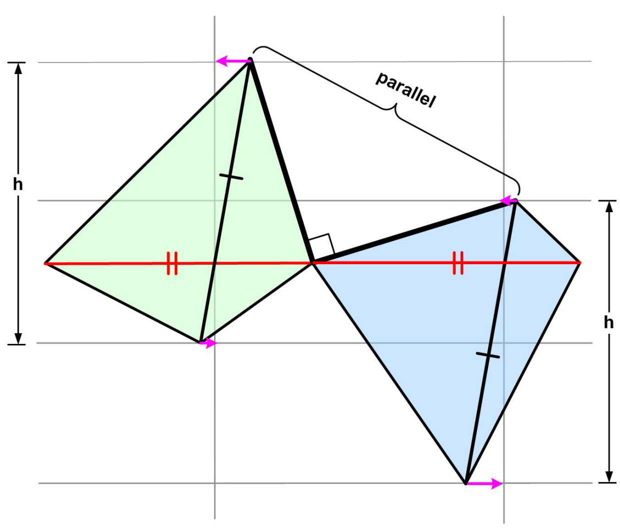


Figure 3 Rotated "Half"

lines are vertical (Figure 4). Recall that the shearing does not change the areas. To make the result completely evident we will shear the green quadrilateral down and the blue quadrilateral up vertically to obtain explicit triangles (Figure 5), which have the same area, since they have the same base and altitudes. Thus the original black line cut the quadrilateral into two equal areas.

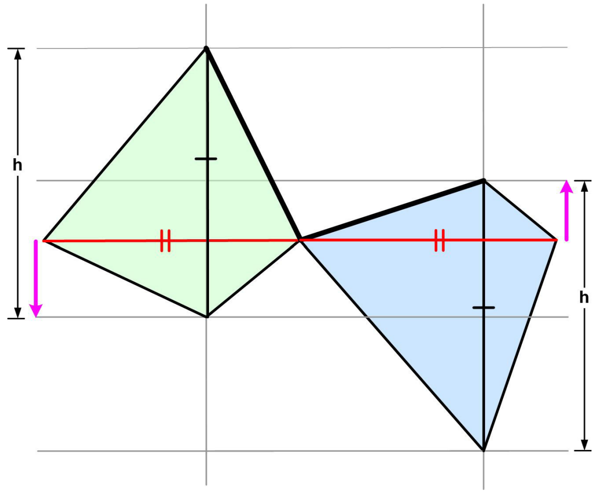


Figure 4 Result of horizontal shearing

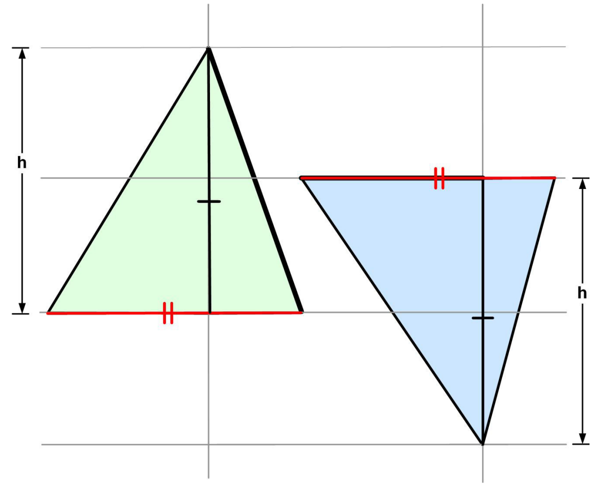


Figure 5 Result of vertical shearing

Quantum Solution

The Quantum solution was not quite pure plane geometry, but involved a bit of trigonometry. Still it implicitly used the shearing property.

In Figure 6, the area of the quadrilateral AECF is equal to

$$\tfrac{1}{2}\, AC \cdot EF \cdot \sin\alpha$$ [1]

and the area of EBDC is

$$\tfrac{1}{2}\, BC \cdot ED \cdot \sin\beta.$$

Now we notice that $AC = BC$, $EF = ED$ and $\sin\alpha = \sin\beta$, because

$$\alpha + \beta = 360° - (\angle ACB + \angle FED) = 180°.$$

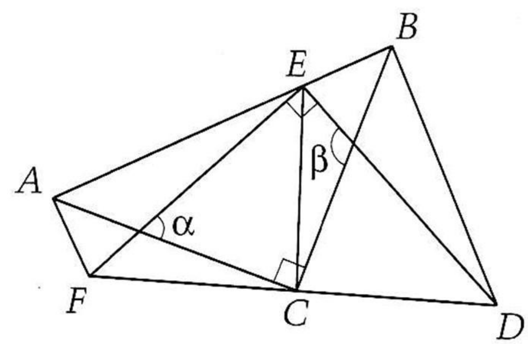


Figure 6 *Quantum* Solution

References

[1] "Playing with Triangles" B159 "Brainteasers" *Quantum* Vol.6, No.2, National Science Teachers Assoc., Springer-Verlag, Nov-Dec 1995. p.11

[1] JOS: This is effectively equivalent to the shearing argument (in trigonometric form).