Ladder Locus Puzzle

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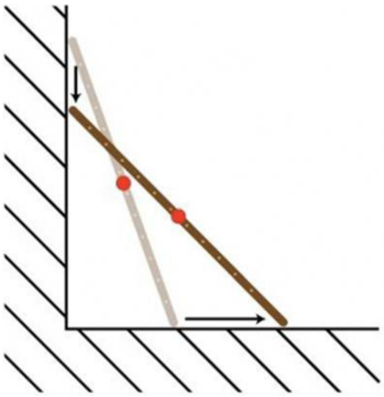
Jim Stevenson



This is a thoughtful puzzle from the Maths Masters team, Burkard Polster (aka Mathologer) and Marty Ross ([1]) as part of their "Summer Quizzes" offerings.

A ladder is leaning against a wall. The base of the ladder starts sliding away from the wall, with the top of the ladder sliding down the wall. As the ladder slides, you watch the red point in the middle of the ladder. What figure does the red point trace? What about other points on the ladder?

My Solution

First Problem. Figure 1 shows a parameterization of the problem with the ladder making an angle θ with the ground and the midpoint of the ladder dividing it into two equal segments of length a. Then it is clear that the coordinates of the midpoint (x, y) are given parametrically with respect to θ as

$$x = a \cos \theta$$
$$y = a \sin \theta$$

which are the parametric equations for a circle. (It is easy to see that $x^2 + y^2 = a^2$.)

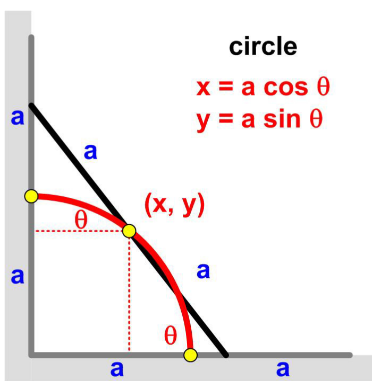


Figure 1

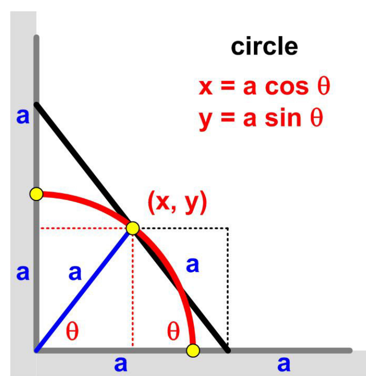


Figure 2

Figure 2 shows the more traditional view of the parametric equations where a is seen as the radius of the circle. The radius makes the same angle θ by virtue of the congruent rectangles via the midpoint properties.

Second Problem. If we choose any other point along the ladder, we proceed similarly. Figure 3 shows a generic situation where the shorter segment of the ladder from the selected point is of length b and the longer segment is of length a. Again the coordinates of the selected point (x, y) are given by the relationship

$$x = b \cos \theta$$
$$y = a \sin \theta$$

which are the parametric equations for an ellipse with semi-major axis a and semi-minor axis b. (Of course when the shorter segment b is at the bottom of the ladder, a and b are switched in the

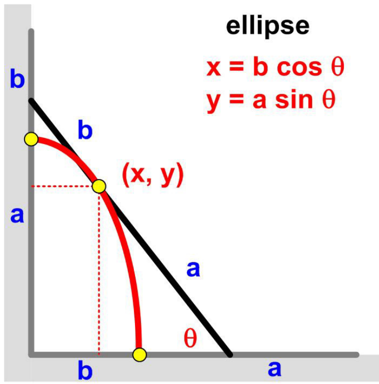


Figure 3

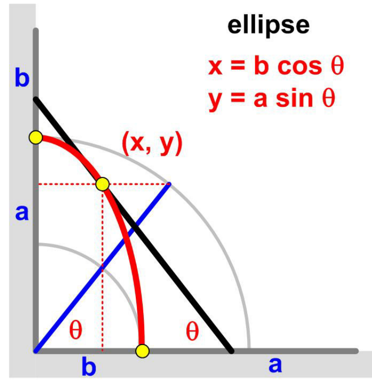


Figure 4

parametric equations.)

And again Figure 4 shows the more traditional view of the elliptical parametric equations, where we have superimposed on the diagram two quarter circles of radius a and radius b. The blue line making an angle θ with the ground intersects these circles at points giving the desired parametric values. The Cartesian coordinate version is easily seen from the parametric equations to be

$$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$$

Comment. As is typical of any problem solving, I did not arrive at the solution to the first problem directly. In fact, I first noticed the situation in Figure 2 where the line for the origin to (x, y) was the same length as half the ladder (via all the congruent figures). That meant (x, y) was always a constant distance from the origin or moving along a circle. The parametric equation representation came about from the solution to the second problem, where an ellipse was anticipated after the solution to the first problem. Again I discovered the relation shown in Figure 4 first (via Visio).

Maths Masters Solution

The Maths Masters' solution keeps to Cartesian coordinate forms for the equations (Figure 5).

Solution: Suppose the red point has coordinates (X, Y), with the base of the ladder a further A units along, and suppose the ladder has length 2. In the diagram below, the two green triangles are equal, and so $A = X$. And, Pythagoras's Theorem applied to the darker triangle gives $A^2 + Y^2 = 1$. So, $X^2+Y^2 = 1$, and we see that the red point moves along a quarter circle.

The endpoints of the ladder obviously describe straight lines. For other points, the diagram is as above, but the triangles are similar rather than equal. Think of a dot splitting the ladder into a lower piece of length L, and an upper piece of length $2 - L$. Then, by Pythagoras, we have $A^2 + Y^2 = L^2$. Using the similar triangles, we also have $A / L = X / (2 - L)$. Solving for A and plugging into the previous equation, we get the equation for an ellipse.

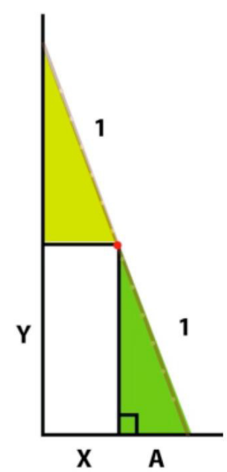


Figure 5

References

[1] Polster, Burkard and Marty Ross, "Maths Challenge 2009, Problem 19", *The Age*, 30 November 2009 (https://www.qedcat.com/summerquizzes/2009%20QUIZ.pdf)