Ubiquitous Radius Problem

28 October 2021

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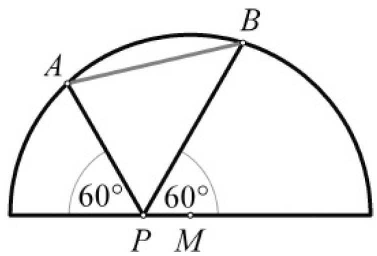

This is an interesting problem from Posamentier and Lehmann's *Mathematical Curiosities* ([1]).

In the figure we have a semicircle with the point *P* randomly placed on the diameter. Points *A* and *B* are situated on the circle such that they form angles of 60° with the diameter as shown in the figure. This problem asks us to show that the length of *AB* is equal to the radius of the semicircle.

My Solution

First, notice that the unlabeled angle at P is also 60° (= 180° – 120°). Now expand the semicircle to a full circle (Figure 1) and drop a perpendicular from the left endpoint A of the blue line. This makes a 30-60 right triangle. Flip this triangle about the diameter of the semicircle, making a straight line chord to the bottom half of the circle. The hypotenuse of the lower right triangle (dashed green line) makes a straight line with the right side of the red triangle (line PB), since 3·60° = 180°. Then 30° is the inscribed angle subtended by the blue line.

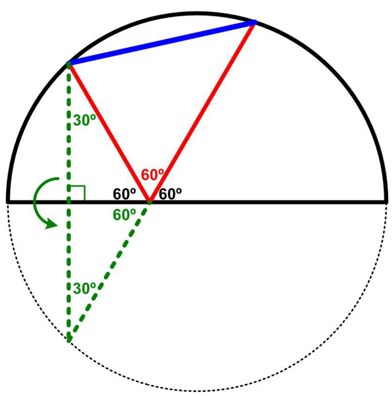

Figure 1

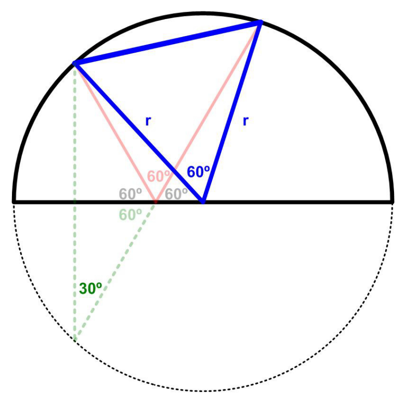

Figure 2

This means the central angle subtended by the original blue line is 60° (Figure 2). But given that it is an isosceles triangle with each side a radius, it must be an equilateral triangle. Therefore the original blue line (AB) has the length of the radius.

Posamentier Solution

This problem appears to be rather complicated, since the result seems counterintuitive. Remember the point P is any point along the diameter, and yet the line segment AB is to be shown to be the same length as the radius. We begin our demonstration by constructing auxiliary lines as shown in Figure 3. Since parallel lines cut off equal arcs along a circle we can conclude—from the parallel lines drawn in the figure—that the arcs AX, BY, and CZ are equal. Therefore we can conclude that the arcs AB and XY are equal, as are their respective chords: AB = XY. However, the chord XY is the third side of an isosceles triangle whose vertex angle ∠XMY = 60°. That makes the triangle XMY equilateral.

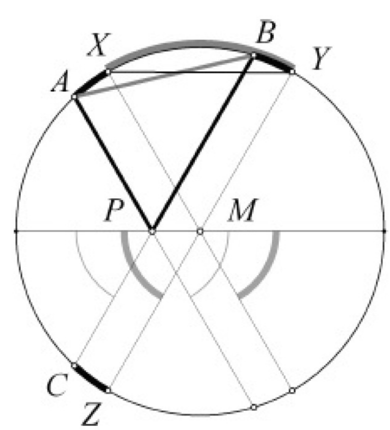

Figure 3

Thus, we have shown that the chord XY is equal to the radius of the circle, since it is equal to XM, and it follows that AB is equal to the radius of the circle as well.

References

[1] Posamentier, Alfred S. and Ingmar Lehmann, *Mathematical Curiosities: A Treasure Trove of Unexpected Entertainments*, Prometheus Books, 2014