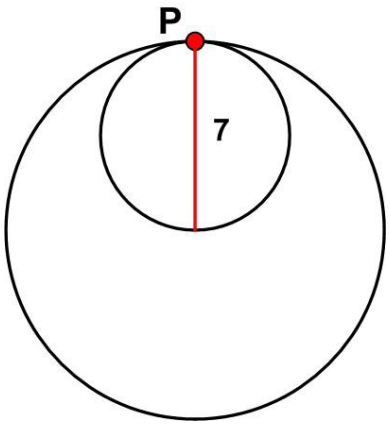
Wandering Epicycle

29 October 2021

Jim Stevenson



Here is an intriguing problem from the 2021 Math Calendar ([1]).

If the smaller circle of diameter 7 rotates without slipping within the larger circle, what is the length of the path of P?

The problem did not state clearly how far the smaller circle should rotate. Its answer implied it should complete just one full (360°) rotation within the larger circle.

Recall that all the answers are integer days of the month.

Solution

The answer is shown in Figure 1-Figure 3 where the smaller circle is shown rotating to the right in the larger circle. A full 360° rotation of the smaller circle takes it to the bottom of the larger circle and the point P has traveled down along the diameter of the larger circle. Thus the length of the path of P is $2 \cdot 7 = 14$.

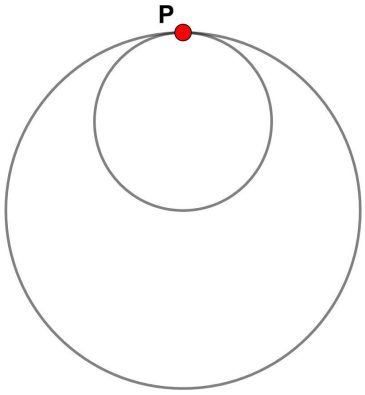


Figure 1

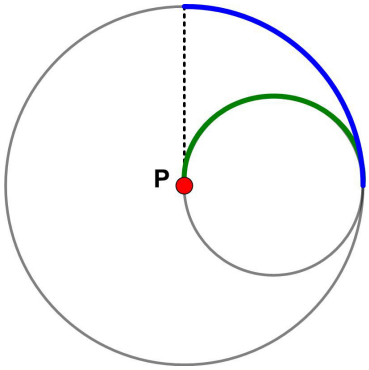


Figure 2

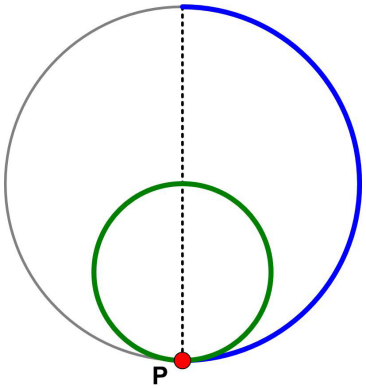


Figure 3

Now the proof of this result is shown in Figure 4. As the smaller circle rotates along the side of the larger, the arclength between its tangent point Q to the larger circle and P is the same as the arclength of the distance traveled along the larger circle.

If we let r be the radius of the smaller circle (= 7/2 in the problem), then the arclength of the traveled distance on the larger circle (sweep of the radius of the larger circle) is $(2r)\theta$. So the arclength between the tangent point Q and P on the smaller circle (sweep of the radius of the smaller circle) must be the same, $r(2\theta)$, which means the smaller circle is rotating counter clockwise twice as fast as it is moving clockwise around the larger circle.

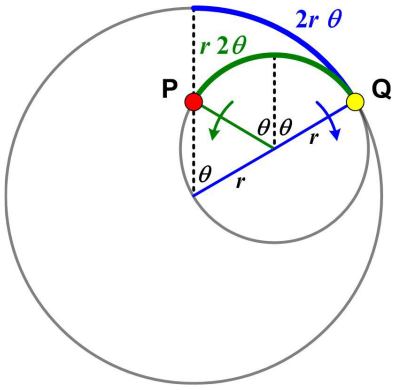


Figure 4

This effectively cancels the lateral motion of P so that it only moves vertically down, namely along the diameter of the larger circle. So when the smaller circle has completed one rotation (P has traveled the full circumference),

the smaller circle has only traveled halfway around the larger circle, and so rests on the bottom (Figure 3).

A nice animated GIF[1] of this behavior is shown in Figure 5.

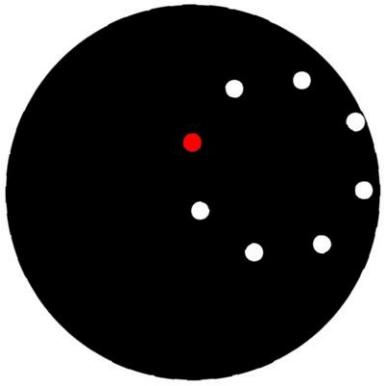


Figure 5 Animated GIF of epicycle motion

References

[1] Rapoport, Rebecca and Dean Chung, *Mathematics 2021: Your Daily epsilon of Math*, Rock Point, Quarto Publishing Group, New York, 2021. January

[1] https://i.imgur.com/DaTi4L9.gif