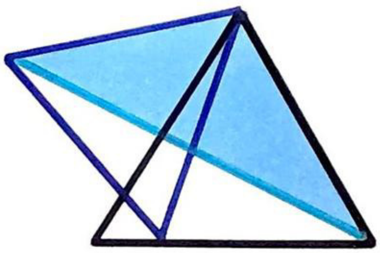
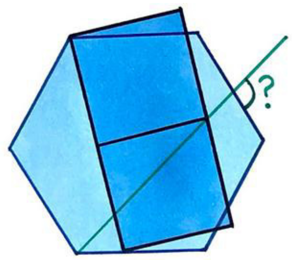
Geometric Puzzle Mindbogglers

27 September 2021

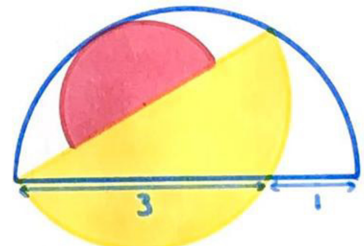
Jim Stevenson



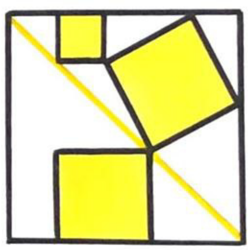
#1 Swinging Triangle. The largest of these two equilateral triangles has area 8. What's the blue area?



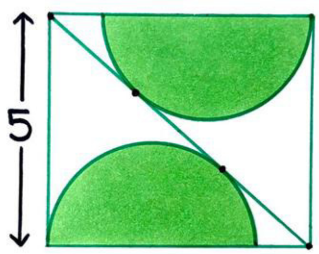
#2 Dancing Rectangle. Two squares and a regular hexagon. What's the angle?



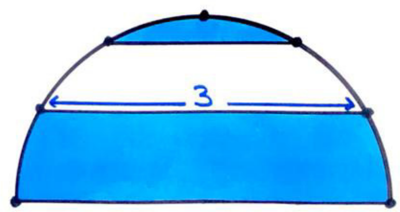
#3 Tipsy Boat. What's the total shaded area?



#4 Tumbling Squares. Four squares. What fraction is shaded?



#5 Floating Bowls. The dots are equally spaced along the rectangle's diagonal. What's the total shaded area?



#6 Cloudy Blue Moon. The points on the circumference are equally spaced. What's the total shaded area?

Here is yet another collection of beautiful, stimulating geometric problems from Catriona Agg (née Shearer).

Solutions

#1 Swinging Triangle[1]

We will consider the canted equilateral triangle as swinging from the top vertex, maintaining its lower vertex along the base of the large equilateral triangle (Figure 1). Notice that as the right (red) side of the triangle moves along the base, the left (blue) side moves at a constant 60° angle with respect to the red side and at the same length as the red side. Therefore, the endpoint of the left blue side also traces a straight line and at an angle of 60° with respect to the base of the large triangle.[2] This means the grey straight line is parallel to the right side of the large equilateral triangle.

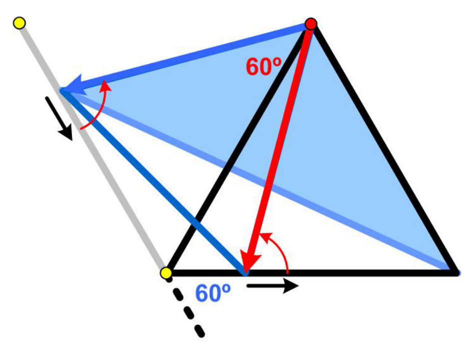


Figure 1

[1] Apr 11, 2021 https://twitter.com/Cshearer41/status/1381192913007341568

[2] JOS: For earlier discussions of this technique see "Curve Making Puzzle" (https://josmfs.net/2020/11/07/curve-making-puzzle/) and "Geometric Puzzle Magnificence #5" (https://josmfs.net/2021/04/10/geometric-puzzle-magnificence/)

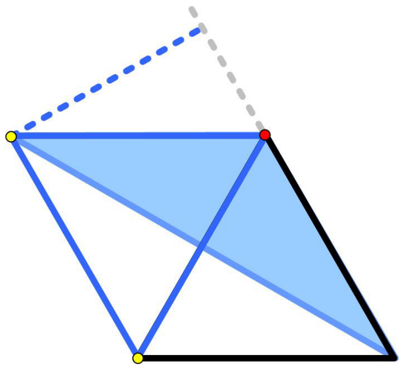


Figure 2

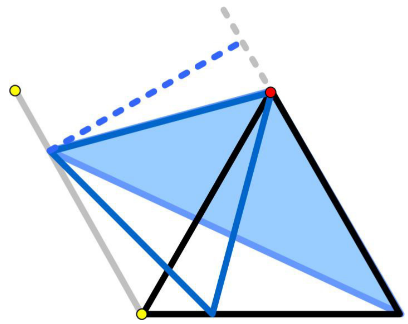


Figure 3

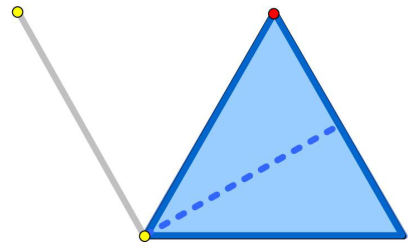


Figure 4

From this we see that the altitude and base of the shaded blue triangle stays constant throughout the swinging motion (Figure 2 - Figure 4). Therefore, the shaded blue triangle has the same area as the large equilateral triangle, namely, 8.

#2 Dancing Rectangle[3]

As in the previous problem, we will move the rectangle so that its top left vertex is fixed on the hexagon and its bottom left vertex moves along the bottom edge of the hexagon (Figure 5). Since the rectangle is composed of two squares, the line in the middle cuts each rectangle in half. As the rectangles tilt, the left endpoint of the midline moves along the horizontal midline of the initial rectangle (Figure 5). Consider a vector drawn from the fixed vertex of the rectangles to the right endpoint of their midlines. This diagonal line and the left and bottom edges of the top square of the initial rectangle make an isosceles right triangle (Figure 6). Thus the diagonal (hypotenuse) makes a 45° angle with the left edge of the square. As the rectangles tilt, the endpoints of the corresponding hypotenuses of the succession of similar right triangles move up a straight line as the left side moves along the horizontal midline (Figure 6). Since the angle between the diagonal and the left edge is 45°, then the tilt of the straight line is also 45°. This tilted line is an extension of the diagonal through the lower initial square.

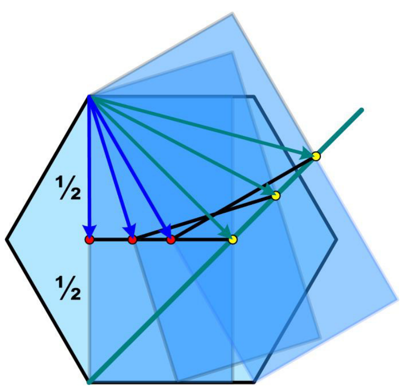


Figure 5

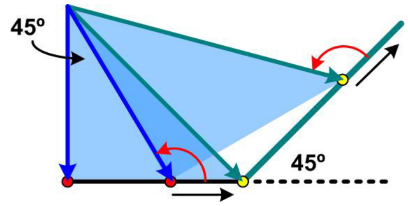


Figure 6

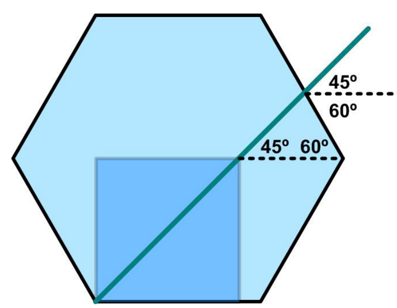


Figure 7

Finally, combining the 45° angle with the angles in a hexagon (Figure 7), we arrive at the desired angle of 105°.

[3] Mar 17, 2021 https://twitter.com/Cshearer41/status/1372184337676955649

#3 Typsy Boat[4]

There are a number of steps in the solution which are shown by the circled numbers in Figure 8.

Step 1. Since the diameter of the large (blue) semicircle is 4, its radius must be 2.

Step 2. Since the radius is 2 and the distance from the left endpoint of the diameter to the intersection of the yellow semicircle with the blue semicircle is 3, that intersection must bisect the radius.

Step 3. Draw a (dashed) line between the two intersection points of the yellow semicircle with the blue semicircle. Together with the line of length 3 and the diameter of the yellow semicircle this dashed line makes a right triangle.

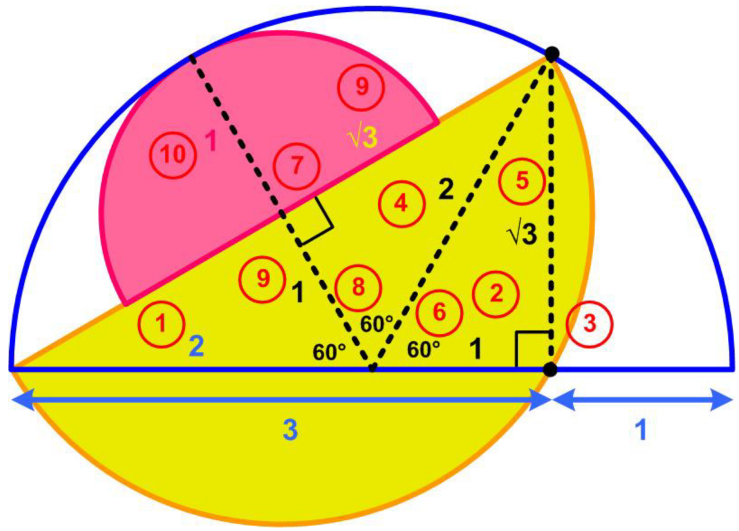


Figure 8

Step 4. Draw a (dashed) radial line to the upper intersection point of the yellow semicircle with the blue semicircle. This makes a right triangle.

Step 5. Therefore the vertical dashed line is of length √3.

Step 6. And the triangle is a 30-60 right triangle with 60° at the vertex at the center of the blue semicircle. This means its supplement angle is 120°.

Step 7. Drop a (dashed) perpendicular bisector of the diameter of the yellow semicircle. It will pass through the center of the blue semicircle (as well as its own center). Therefore it is a radius of the blue semicircle and of length 2.

Step 8. The two right triangles lying on the diameter of the yellow semicircle have three equal sides, and so are congruent. Since the vertex angles at the center of the blue semicircle are equal, they must be 120°/2 = 60° each. Therefore they are also 30-60 right triangles.

Step 9. Since the right-hand of this pair of triangles has a hypotenuse in common with the right-most 30-60 triangle, these two triangles are also congruent. Therefore, they both have sides 1 and √3. This makes √3 the length of the yellow semicircle radius.

Step 10. Since the perpendicular bisector is a radius of the blue semicircle, it is perpendicular to the tangent to the blue semicircle, and so also perpendicular to the tangent to the red semicircle at the same point. Therefore it lies along the radius of the red semicircle, which must be of length 2 – 1 = 1.

Conclusion. We now have the radius of the red semicircle is 1 and that of the yellow semicircle is √3. So their combined areas are $(\pi 1^2 + \pi(\sqrt{3})^2)/2 = 2\pi$.

#4 Tumbling Squares[5]

We parameterize the problem as shown in Figure 9. The side of the bottom square is of length a, that of the top small yellow square is of length b, and that of the canted yellow square is c. Now since

[4] Mar 6, 2021 https://twitter.com/Cshearer41/status/1368140144608501763
[5] Mar 28, 2021 https://twitter.com/Cshearer41/status/1376094496379768838

both the “a” square and “b” square have vertices on the diagonal yellow line, they are respectively a distance a and b from the vertical sides of the enclosing black square.

By symmetry the “c” square is inscribed in a larger (red dashed) square. The green right triangles all have the same angles and so are similar. But their hyptenuses are all sides of the square and so equal. Therefore, the green right triangles are all congruent and so have the corresponding sides a, b, and c, giving $a^2 + b^2 = c^2$.

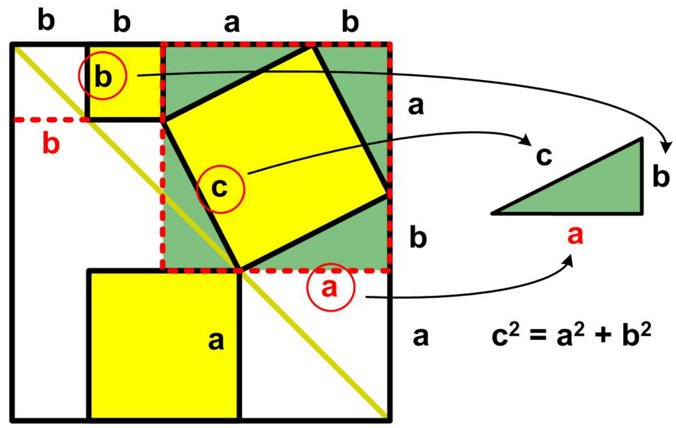


Figure 9

The large black square has sides 3b + a = 2a + b, which implies 2b = a. Therefore its area is

$$(2a + b)^2 = (5b)^2 = 25b^2$$

Then the sum of the yellow squares’ areas is

$$a^2 + b^2 + c^2 = 2(a^2 + b^2) = 2((2\ b^2) + b^2) = 10\ b^2$$

Hence, the ratio of the yellow squares’ areas to that of the enclosing black square is

$$10\ b^2/25b^2 = 2/5$$

#5 Floating Bowls[6]

The diagonal of the rectangle and its left edge are tangent to the semicircle and so each perpendicular to a radius (Figure 10). Therefore we have two right triangles with a pair of equal sides. Therefore the triangles are congruent and all corresponding sides equal. Thus the segment of the diagonal up to the perpendicular radius is of length 5. And so each segment is of length 5/2.

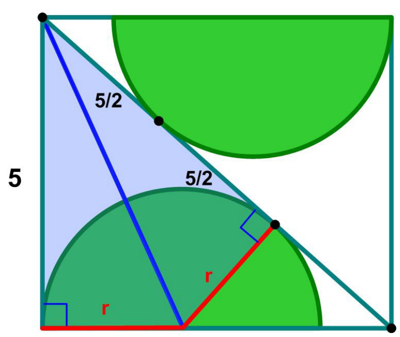


Figure 10

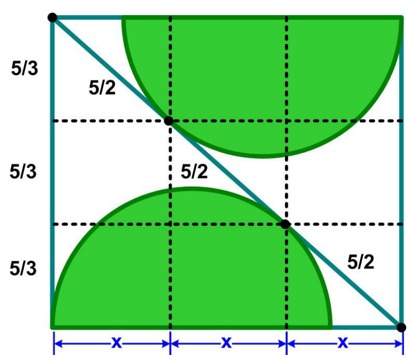


Figure 11

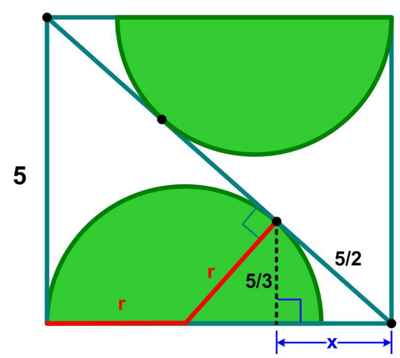


Figure 12

Superimpose an equally-spaced grid on the figure as shown in Figure 11. We wish to solve for x. From Figure 12 we have $x^2 + (5/3)^2 = (5/2)^2$. Therefore, $x = (5/6)\sqrt{5}$. Given the right triangle involving r, we have

$$(5/2)^2 + r^2 = (3x - r)^2 = ((5/2)\sqrt{5} - r)^2 = (5/2)^2 5 - 5\sqrt{5}r + r^2.$$

so
$$5\sqrt{5}r = 4(5/2)^2 \quad \text{or} \quad r = \sqrt{5}.$$

[6] Mar 16, 2021 https://twitter.com/Cshearer41/status/1371806805773463552

Thus, the area of the combined semicircles is $\pi(\sqrt{5})^2 = 5\pi$.

#6 Cloudy Blue Moon[7]

First, divide the semicircle into 6 sectors, each of area S, determined by the equally-spaced points given in the problem (Figure 13). Next, show the 4 congruent 30-60 right triangles, each of area T, determined by the points on the semicircle, based on symmetries about the vertical radius and a radii at 45° and 135° (Figure 14). Finally, note that the lower right triangles determine rectangles so that the upper halves are also congruent 30-60 right triangles (Figure 15).

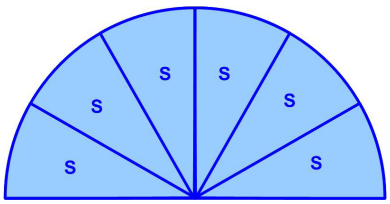


Figure 13

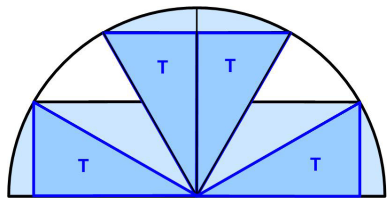


Figure 14

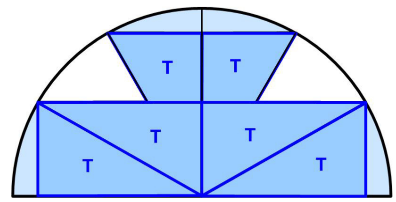


Figure 15

Since the central horizontal line is of length 3, the long edge of the 30-60 right triangle is 3/2. So we have the situation shown in Figure 16 for r, which yields

$$(3/2)^2 = r^2 - (r/2)^2 = 3r^2/4 \quad \text{or} \quad r = \sqrt{3}.$$

Then the area of the sector is $S = ½\, r^2\, \pi/6 = \pi/4$ and the area of the semicircle is $C = ½\, \pi\, r^2 = 3\pi/2$.

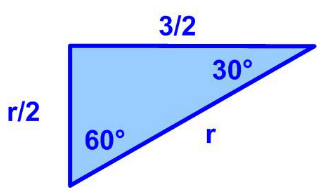


Figure 16

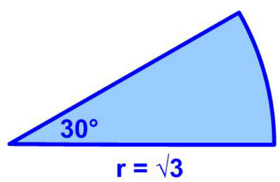


Figure 17

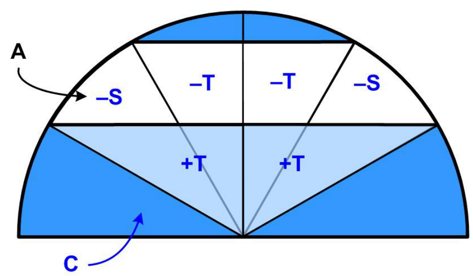


Figure 18

So the final area A of the shaded region of the semicircle is computed using the decomposition shown in Figure 18:

$$A = C - 2S - 2T + 2T = 3\pi/2 - 2\pi/4 = \pi$$

(Notice we didn't have to compute the areas of the triangles, since they canceled.)

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[7] Mar 14, 2021 https://twitter.com/Cshearer41/status/1371073982049435649