Surprising Identity

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Here is surprising problem from the 1875 *The Analyst* ([1])

81. BY G. W. HILL, NYACK TURNPIKE, N. Y. — "Prove that, *identically*,

$$\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\ldots+\frac{1}{2n}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots-\frac{1}{2n}.\text{"}$$

By "identically" the proposer means for all $n = 1, 2, 3, \ldots$.

My Solution

When given a statement we wish to prove for all n ($n = 1, 2, 3, \ldots$), we should immediately think of proceeding by mathematical induction. Mathematical induction was discussed in my "Power of 2 Problem" of 28 December 2018.[1] As a quick review, the idea of mathematical induction, as explained in that posting, is the following:

Principle of Mathematical Induction:

Given: (i) Statement P(n) associated with each natural number $n = 1, 2, 3, \ldots$

(ii) P(1) is true.

(iii) For all natural numbers $k = 1, 2, 3, \ldots$, if P(k) is true, then P(k+1) is true.

Then: For all natural numbers $n = 1, 2, 3, \ldots$, P(n) is true.

The strange thing about this approach is that in place of the original infinite set of statements to prove, P(n), we try to prove another infinite set of statements, S(k) = "P(k) $\Rightarrow$ P(k+1)", where "A $\Rightarrow$ B" means *A implies B* or *if A, then B*. For this to work, it must be easier to prove the S(k) than the P(n). An analogy might be we wish to prove we can knock down a whole set of dominoes with one push. P(n) might be "domino n will fall". Now if we check that for every pair of dominoes, if the first one falls the next one will, say by separating them by half their height, then if we make the first domino fall, we can be assured they all will fall.[2]

At first, I found the identity hard to believe, so I plugged in a few numbers to check and was surprised that it worked. So let P(n) represent the identity. Notice that there are twice as many terms on the right as on the left. Then P(1) is

$$\frac{1}{1+1}=1-\frac{1}{2} \text{ or } ½ = ½ ,$$

which is true. Now assume P(k) is true, that is,

[1] http://josmfs.net/2018/12/28/power-of-2-problem/

[2] See many examples: https://www.google.com/search?q=Domino+falling+gif. A nifty variation on the falling dominoes effect is this video of a brick double-domino effect (https://www.youtube.com/watch?v=-BTWiZ7CYoI). Matt Parker at Stand-up Maths has a fascinating 20 Mar 2017 posting "Brick double-domino effect explained" (https://www.youtube.com/watch?v=EYkBctqyKic)

$$\frac{1}{k+1}+\frac{1}{k+2}+\frac{1}{k+3}+\ldots+\frac{1}{2k}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots-\frac{1}{2k} \qquad (1)$$

is true. Then consider the left side of P(k+1). By equation (1) (the fact that P(k) is true)

$$\boxed{\frac{1}{k+2}+\frac{1}{k+3}+\ldots+\frac{1}{2k}}+\frac{1}{2k+1}+\frac{1}{2k+2}=$$

$$\boxed{1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots-\frac{1}{2k}-\frac{1}{k+1}}+\frac{1}{2k+1}+\frac{1}{2k+2} \qquad (2)$$

But

$$\frac{1}{2k+2}-\frac{1}{k+1}=\frac{1}{2}\frac{1}{k+1}-\frac{1}{k+1}=-\frac{1}{2(k+1)}$$

Therefore, equation (2) becomes

$$\frac{1}{(k+1)+1}+\frac{1}{(k+1)+2}+\frac{1}{(k+1)+3}+\ldots+\frac{1}{2(k+1)}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots-\frac{1}{2(k+1)-2}+\frac{1}{2(k+1)-1}-\frac{1}{2(k+1)}$$

which is the statement that P(k+1) is true. Hence, for all k = 1, 2, 3, …, P(k) $\Rightarrow$ P(k+1). Therefore, the induction hypotheses are satisfied, and so the identity P(n) is true for all n.

Analyst Solution

Of course, *The Analyst* had a slick solution that did not involve mathematical induction.

SOLUTION BY ARTEMAS MARTIN, EIRE, PA.

Let
$$P=1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\ldots+\frac{1}{2n-1} \qquad (3)$$

and
$$Q=\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\ldots+\frac{1}{2n} \qquad (4)$$

By subtraction,
$$P-Q=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots-\frac{1}{2n} \qquad (5)$$

By addition,
$$P+Q=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{2n} \qquad (6)$$

Subtracting twice (4) from (6),

$$P-Q=\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\ldots+\frac{1}{2n}$$

References

[1] *The Analyst: A monthly journal of pure and applied mathematics*, Vol. 2, No. 4 (Des Moines) (Jul., 1875), p. 128. (https://www.jstor.org/stable/2635496)

[2] ———, Vol. 2, No. 5 (Sep., 1875), p. 158 (https://www.jstor.org/stable/2635943)