A Self-Characterizing Figure

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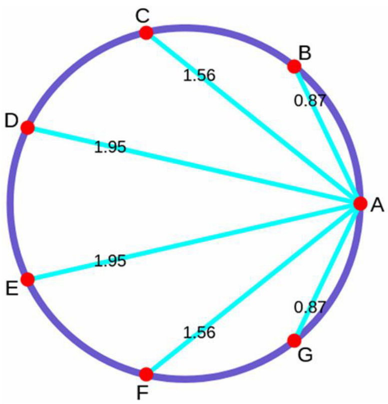
Jim Stevenson



Futility Closet describes a result that is startling, amazing, and mysterious ([1]).

This is pretty: If you choose $n > 1$ equally spaced points on a unit circle and connect one of them to each of the others, the product of the lengths of these chords equals n.

The *Futility Closet* posting includes an interactive display using Wolfram Technology by Jay Warendorff ([3]) that let's you select different n and see the results. It also includes a reference to a paper that proves the result ([2]); only the paper uses residue theory from complex variables, which seems a bit over-kill, though slick, for such a problem. I found a simpler route.

Solution

It is true that the problem immediately suggests looking at complex variables. If we denote the n^{th} roots of unity $z_k = e^{2\pi ki/n}$, where $k = 0, 1, \ldots, n-1$, by vectors from the center to the boundary of the unit circle (Figure 1), then the lengths of the differences $r_k = |\, z_k - z_0 \,|$ for $k = 1, 2, \ldots, n-1$, represent the lengths of the chords of interest.

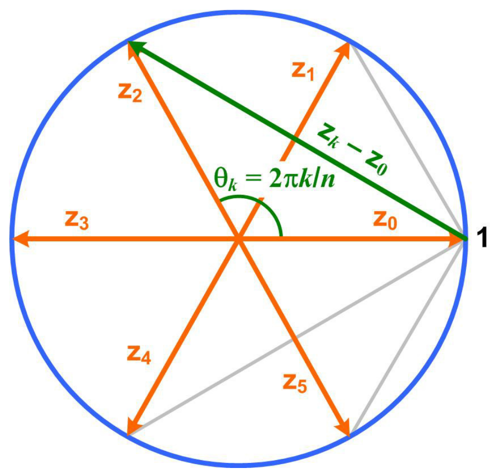


Figure 1 6th Roots of Unity

We will use the property of complex variables that the modulus (length) of the product of two complex numbers u, w is the product of their moduli, that is,

$$|u\, w| = |u|\, |w|$$

Having all the roots of unity and thinking about the differences between roots, we immediately think of the expansion

$$0 = z^n - 1 = (z - z_{n-1})\,(z - z_{n-2})\,\ldots\,(z - z_0)$$

Now

$$z^n - 1 = (z - 1)(z^{n-1} + z^{n-2} + \ldots\, z + 1)$$

Therefore, since $z_0 = 1$,

$$z^{n-1} + z^{n-2} + \ldots\, z + 1 = (z - z_{n-1})\,(z - z_{n-2})\,\ldots\,(z - z_1)$$

and

$$n = |z_0^{n-1} + z_0^{n-2} + \ldots\, z_0^1 + 1| = |z_0 - z_{n-1}|\,|z_0 - z_{n-2}|\,\ldots\,|z_0 - z_1| = r_1\, r_2\, r_3\, \ldots\, r_{n-1}$$

which is what we wanted to show.

Again mysteries vanish when an underlying mathematical pattern is revealed.

References

[1] "A Self-Characterizing Figure", *Futility Closet*, 17 March 2021 (https://www.futilitycloset.com/2021/03/17/a-self-characterizing-figure/)

[2] Andre P. Mazzoleni and Samuel Shan-Pu Shen, "The Product of Chord Lengths of a Circle," *Mathematics Magazine* 68:1 [February 1995], 59-60. (https://shen.sdsu.edu/pdf/mazz_matma_1995.pdf)

[3] Jay Warendorff, Demonstration with Wolfram Technology (https://demonstrations.wolfram.com/AProductOfChordLengthsInACircle/#embed)