Rotating Plane Problem

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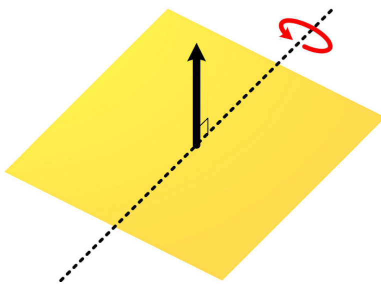

Here is another challenging problem from the first issue of the 1874 *The Analyst* ([2]), which also appears in Benjamin Wardhaugh's book ([1]).

3. If a line make an angle of 40° with a fixed plane, and a plane embracing this line be perpendicular to the fixed plane, how many degrees from its first position must the plane embracing the line revolve in order that it may make an angle of 45° with the fixed plane?

—Communicated by Prof. A. Schuyler, Berea, Ohio.

Part of the challenge is to construct a diagram of the problem. I used techniques for a solution that were barely in use when this problem was posed in 1874. The contrast between then and now is most revealing.

My Solution

Figure 1 represents a vector-based description of the problem. The blue plane is the "fixed plane" of the problem, the black dashed line is the line that makes a 40º angle with the fixed plane. The yellow plane is the plane containing the line and perpendicular to the fixed plane. This plane is rotated about the line through an angle θ so that it makes a 45º angle with the fixed plane.

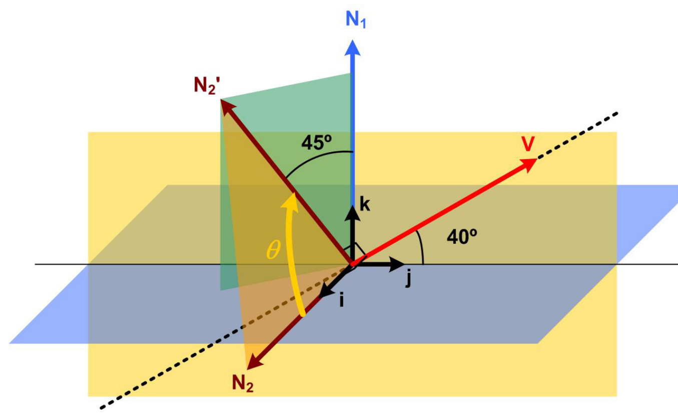

Figure 1 Vector Description (all unit vectors)

But all these statements are represented by angles with respect to particular vectors associated with the line and planes. Recall that a line is uniquely determined by a unit vector giving its direction and a point on the line. Similarly, a plane is uniquely determined by a unit vector giving its "direction" and a point on the plane. The unit vector for a plane is the normal or perpendicular to the plane. These vectors represent direction up to 180º (plus or minus the vector). The normal determining the direction of a plane reminded me of the plasterer's tool (which I recently learned is called a hawk) that holds the plaster for spreading on a wall (Figure 2). Clearly grabbing the handle allows the plasterer to orient the plate in any desired direction. So, all the statements about planes and lines in the problem are represented by equivalent statements about their associated unit vectors. (For clarity, these vectors have not been shown in the figure to be the same length as the coordinate system unit vectors **i**, **j**, **k**, since then they would be indistinguishable.)

Figure 2 Plasterer's Hawk

$\mathbf{N_1}$ is the normal to the fixed plane and coincides with the unit vector $\mathbf{k}$ along the z-axis in our choice of coordinate system. $\mathbf{V}$ is the unit vector along the line making a 40º angle to the fixed plane, and so is given by

$$\mathbf{V} = \cos 40^\circ\, \mathbf{j} + \sin 40^\circ\, \mathbf{k}$$

$\mathbf{N_2}$ is the normal to the plane containing the line and perpendicular to the fixed plane, and so coincides with the unit vector $\mathbf{i}$. Finally, $\mathbf{N_2'} = a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k}$ is $\mathbf{N_2}$ rotated θ degrees about $\mathbf{V}$ to make an angle of 45º with $\mathbf{N_1} = \mathbf{k}$.

All the geometric relationships in the problem become translated into equations involving these vectors. The fact that $\mathbf{V}$ is in the plane determined by $\mathbf{N_2}$ means it is perpendicular to $\mathbf{N_2}$, and the rotated $\mathbf{N_2'}$ as well. Therefore,

$$0 = \mathbf{N_2} \cdot \mathbf{V} = (\mathbf{i}) \cdot (\cos 40^\circ\, \mathbf{j} + \sin 40^\circ\, \mathbf{k})$$

and
$$0 = \mathbf{N_2'} \cdot \mathbf{V} = (a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k}) \cdot (\cos 40^\circ\, \mathbf{j} + \sin 40^\circ\, \mathbf{k})$$

so
$$0 = b \cos 40^\circ + c \sin 40^\circ \tag{1}$$

Furthermore, recall the geometric form of the dot and cross product of vectors: for arbitrary vectors $\mathbf{u}$, $\mathbf{w}$, separated by an angle θ,

$$\mathbf{u} \cdot \mathbf{w} = |\mathbf{u}|\,|\mathbf{w}| \cos\theta \quad \text{and} \quad \mathbf{u} \times \mathbf{w} = |\mathbf{u}|\,|\mathbf{w}| \sin\theta\, \mathbf{n}, \tag{2}$$

where $|\mathbf{u}|$ is the length of the vector $\mathbf{u}$, and $\mathbf{n}$ is a unit vector perpendicular to $\mathbf{u}$ and $\mathbf{w}$ following the right-hand rule. Then the fact that the angle between the rotated $\mathbf{N_2'}$ and $\mathbf{N_1}$ is 45º means

$$1 \cdot 1 \cdot \cos 45^\circ = \mathbf{N_2'} \cdot \mathbf{N_1} = (a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k}) \cdot (\mathbf{k}) = c$$

implies
$$c = 1/\sqrt{2} \tag{3}$$

Therefore, from equation (1) we have

$$b = -\tan 40^\circ / \sqrt{2} \tag{4}$$

Now, combining the geometric and coordinate form of the cross product (equations (2)) and the fact that $\mathbf{V}$ is the unit vector perpendicular to both $\mathbf{N_2}$ and $\mathbf{N_2'}$, we have

$$\sin\theta\, \mathbf{V} = \mathbf{N_2'} \times \mathbf{N_2} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & b & c \\ 1 & 0 & 0 \end{vmatrix} = c\,\mathbf{j} - b\,\mathbf{k}$$

and so

$$\sin\theta\, \mathbf{V} \cdot \mathbf{V} = \mathbf{N_2'} \times \mathbf{N_2} \cdot \mathbf{V} = (c\,\mathbf{j} - b\,\mathbf{k}) \cdot (\cos 40^\circ\, \mathbf{j} + \sin 40^\circ\, \mathbf{k}),$$

which implies, from (3) and (4),

$$\sin\theta = (\cos 40^\circ + \tan 40^\circ \sin 40^\circ) / \sqrt{2} = 1 / (\cos 40^\circ \sqrt{2})$$

or

$$\theta = \sin^{-1} [1 / (\cos 40^\circ \sqrt{2})] = 67.37796369 = 67^\circ\, 22'\, 40.67''$$

Comment. This may seem complicated at first, but the vectors make the translation of the problem into a visual description quit easy. The geometric and coordinate properties of vectors and their arithmetic are used to further translate the constraints of the problem. Then solving the problem just becomes turning the crank on vector properties. See how different this is from the 1874 approach to the solution below.

The Analyst Solution

Again I found the solution in a copy of the first year of *The Analyst* at JSTOR ([3]). My commentary will follow.

SOLUTION BY PROF. E. W. HYDE, CHESTER, PA.

Let b = angle between line and plane; C = angle through which plane through line must revolve, and B = angle plane makes with fixed plane; then we have at once, by Napier's formulae,

$$\cos B = \cos b \sin C;$$

$$\therefore \sin C = \cos B / \cos b,$$

and if $b = 40°$ and $B = 45°$,

$$\sin C = \cos B / \cos b = 1 / \sqrt{2} \cos 40°;$$

$$\therefore C = 67° \; 22' \; 40''.$$

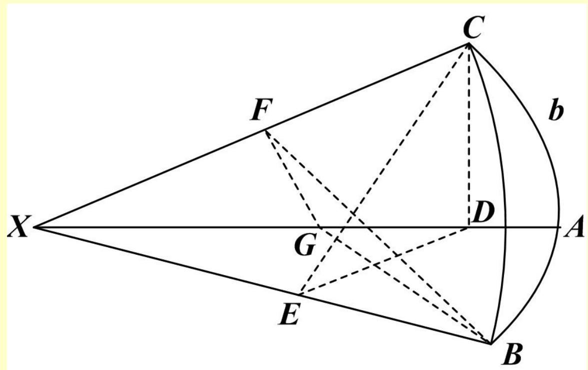

Figure 3

[All the solutions of No. 3 which have been received, except Mr. Salmon's, are analogous to the above.]

SOLUTION BY S. W. SALMON, MOUNT OLIVE, N. J.

Take the fixed plane as the horizontal plane of projection (H). Let the given line CA make an angle θ with H, and take the vertical plane through this line as the vertical plane of projection (V). Let G be the point in which CA pierces H. Let γ be the angle through which the vertical plane through GA has to revolve in order to make an angle ϕ with H, and let β be the angle which the horizontal trace of this plan makes with the ground line. Draw CF perpendicular to H, and let it be the axis of a cone with a circular right section whose vertex is C and whose elements make an angle ϕ with H. Through CA pass a plane tangent to this cone; the tangent plane will then make an angle ϕ with H; CA is the vertical trace of this plane. In order to find the horizontal trace pass a plane parallel to H through A; it cuts a circle from the cone and a line from the plane tangent to the circle. ED is the horizontal projection of this line, and CG, drawn parallel to ED, is the horizontal trace of the plane. Through A pass a plane perpendicular to CA, AH is its vertical and GH its horizontal trace; it cuts a line from the plane GCA, the position of which when revolved around GH into the horizontal plane is GK. The angle $GKH = \gamma$.

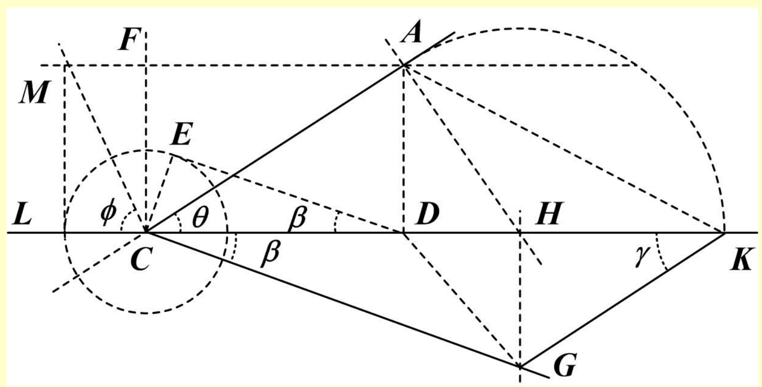

Figure 4

$$LM = AD = CD \operatorname{tang} \theta$$

and $$LC = CE = LM / \operatorname{tang} \phi = CD \operatorname{tang} \theta / \operatorname{tang} \phi = CD \sin \beta,$$

whence $$\beta = \sin^{-1} (\operatorname{tang} \theta / \operatorname{tang} \phi);$$

$$\text{tang}\,\gamma = \frac{GH}{HK} = \frac{CH\ \text{tang}\beta}{CH\sin\theta} = \frac{\text{tang}\left(\sin^{-1}\dfrac{\text{tang}\theta}{\text{tang}\phi}\right)}{\sin\theta}$$

$$\therefore\ \ \gamma = \text{tang}^{-1}\left[\frac{\text{tang}\left(\sin^{-1}\dfrac{\text{tang}\theta}{\text{tang}\phi}\right)}{\sin\theta}\right]$$

When $\theta = 40º$ and $\phi = 45º$, $\gamma = 67º\ 22'\ 41''$.

[This question admits of still another solution, as follows: In the figure to Prof. Hyde's solution, above, draw *CD* perpendicular to *AX* and *DE* perpendicular to *BX*. Then, because the ∠*CED* = 45º, *DE* = *CD*. ∴ (if *CX* = 1) CE= $\sqrt{2}$ sin 40º, and *EX* = $\sqrt{1 - 2\sin^2 40º}$. Make *FX* = *EX*; then is *FG* = tan 40° $\sqrt{1 - 2\sin^2 40º}$. Join *BG* and *BF*. Then is BGF a right angled triangle, right angle at *G*, and *BF* = *CE*. ∴ we have *BF*: *FG* :: radius : cos of the required angle, or, $\sqrt{2}$ sin 40º : tan 40° $\sqrt{1 - 2\sin^2 40º}$ :: 1 : cos *GFB* = 67º 22' 41", very nearly.]

Comments

First, I have to admit I think I see the geometric relationships of the problem in the first diagram, but I am so wedded to employing vectors instead of visualizing intersecting planes that it is hard to tell. However, the second figure and solution are so opaque for me that I didn't bother to try to follow it. The first solution is instantly done via "Napier's formulae"—something I don't immediately recall. These must come from spherical geometry or spherical trigonometry, which was probably well-known and readily at hand in 1874. But now we can handle spherical geometry with vectors and not have to remember a lot of additional formulas beyond basic plane geometry and plane trigonometry.

Historical Perspective. So why did the solutions to the problem rely on spherical geometry rather than vectors? The simple answer is that vectors barely existed in 1874 and spherical geometry and trigonometry were venerable subjects that had dominated mathematics for over 2000 years as the main world-wide tool to study astronomy.

Morris Kline in his history of mathematics ([4]) describes the dominance of spherical trigonometry and geometry for studying astronomy down through the ages:

> Entirely new in the Alexandrian Greek quantitative geometry was trigonometry, a creation of Hipparchus, Menelaus, and Ptolemy. This work was motivated by the desire to build a quantitative astronomy that could be used to predict the paths and positions of the heavenly bodies and to aid the telling of time, calendar-reckoning, navigation, and geography.
>
> The trigonometry of the Alexandrian Greeks is what we call spherical trigonometry though, as we shall see, the essentials of plane trigonometry were also involved. Spherical trigonometry presupposes spherical geometry, for example the properties of great circles and spherical triangles, much of which was already known; it had been investigated as soon as astronomy became mathematical, during the time of the later Pythagoreans. Euclid's *Phaenomena*, itself based on earlier work, contains some spherical geometry. Many of its theorems were intended to deal with the apparent motion of the stars. … [p.119]
>
> We should note that trigonometry was created for use in astronomy; and, because spherical trigonometry was for this purpose the more useful tool, it was the first to be developed. The use of plane trigonometry in indirect measurement and in surveying is foreign to Greek mathematics.

> This may seem strange to us, but historically it is readily understandable, since astronomy was the major concern of the Greek mathematicians. … [p.125-6]
>
> Until 1450, trigonometry was largely spherical trigonometry; surveying continued to use the geometric methods of the Romans. About that date plane trigonometry became important in surveying, though Leonardo of Pisa in his *Practica Geometriae* (1220) had already initiated the method. [p.237]

So what about vectors? They arose in the mid-19th century as a by-product of the new number system called quaternions introduced by William Rowan Hamilton as a means to describe rotations in three-dimensional space, as complex numbers were used to describe two-dimensional rotations. Similar ideas were developed by Hermann Grassmann, but they did not flow out of an extension to the complex number system like Hamilton's. A thorough history of the development of vector analysis can be found in Michael J. Crowe's *A History of Vector Analysis: The Evolution of the Idea of a Vectorial System* (1967), which is summarized in a talk he gave at the University of Louisville in 2002 ([5]). At one point he gives a helpful timeline:

> It is useful to analyze the development of modern vector analysis in terms of three periods, the first extending up to 1865, by which time the two main traditions, the Hamiltonian quaternionic and the Grassmannian tradition had arisen. The second or middle period runs from about 1865 to about 1880. By the beginning of this period, Hamilton (because of his death) and Grassmann (who concentrated on other areas) had ceased to be major contributors. Other mathematicians had gradually assumed positions of leadership. In the third period, which began around 1880, the modern system of vector analysis came into existence through the work of Josiah Willard Gibbs and Oliver Heaviside and by 1910 had established itself as the dominant system, although not without a struggle against the Hamiltonian and Grassmannian systems.

So it is clear that this new system of vectors did not really gain prominence until the beginning of the 20th century, well after the time of *The Analyst* in 1874. It is rather startling how quickly a long-standing area of mathematics, such as spherical geometry and trigonometry, can be transformed by a new system of mathematics, such as vectors, and become practically obsolete in its original form.

References

[1] Wardhaugh, Benjamin, ed., *A Wealth of Numbers: An Anthology of 500 Years of Popular Mathematics Writing*, Princeton University Press, Princeton, New Jersey, 392pp., 2012. pp.71-73

[2] *The Analyst: A monthly journal of pure and applied mathematics*, Vol. 1, No. 1 (Des Moines) (Jan., 1874), p. 15. (https://www.jstor.org/stable/2635496)

[3] ———, Vol. 1, No. 3 (Mar., 1874), pp. 48-50 (https://www.jstor.org/stable/2636173)

[4] Kline, Morris, *Mathematical Thought from Ancient to Modern Times*, Oxford University Press, New York, 1972 (https://archive.org/details/MathematicalThoughtFromAncientToModernTimes) pp.119-127, 237-240.

[5] Crowe, Michael J., "A History of Vector Analysis", Talk at Department of Mathematics University of Louisville, Autumn Term, 2002. Based on his book, *A History of Vector Analysis: The Evolution of the Idea of a Vectorial System* (Notre Dame, Indiana: University of Notre Dame Press, 1967); paperback edition with a new preface (New York: Dover, 1985); another edition with new introductory material (New York: Dover, 1994).