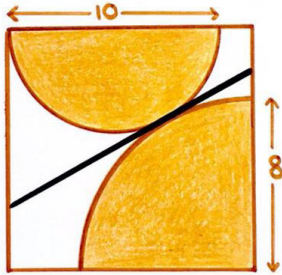


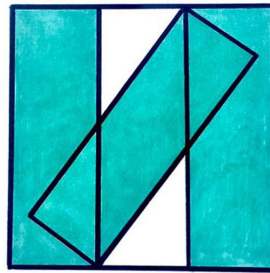
# Geometric Puzzle Magnificence

2 December 2020

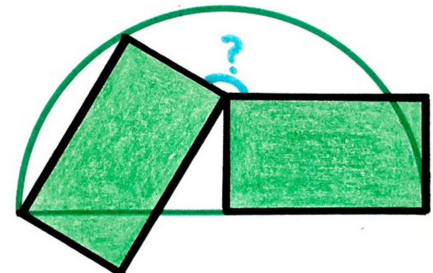
Jim Stevenson



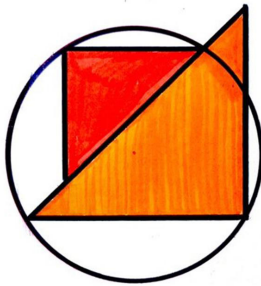
#1 A semicircle and quarter circle inside a square. How long is the black tangent line?



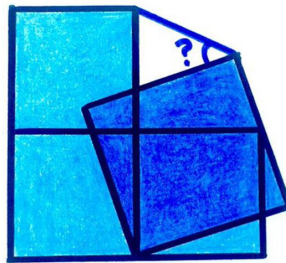
#2 The three green rectangles are congruent. What fraction of the square do they cover?



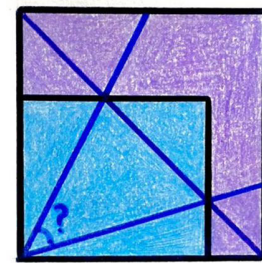
#3 The two rectangles are congruent. What's the angle?



#4 Two half squares, and a circle of radius 1. What's the total shaded area?



#5 Four squares. What's the angle?



#6 The blue rectangle covers half of the square's area. What's the angle?

Here is yet another collection of beautiful geometric problems from Catriona Agg (née Shearer). For some reason I found these a bit more challenging than the previous ones. Some of them required more time to “see” the breakthrough.

## Solution to Problem #1<sup>1</sup>

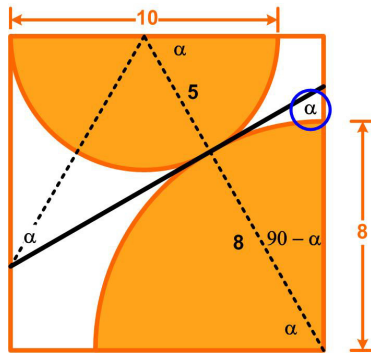


Figure 1

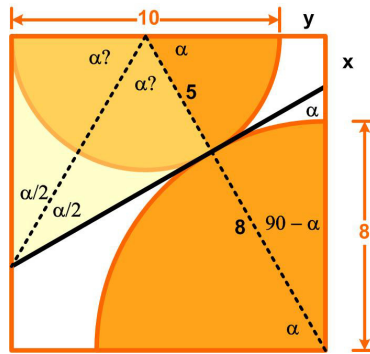


Figure 2

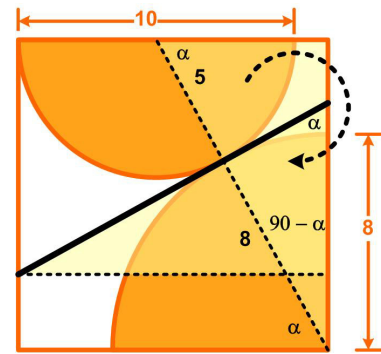


Figure 3

Since I use Viseo a lot to solve these geometric problems, I thought it might be instructive to show how Viseo led me down a blind alley. Figure 1 shows how I annotated the original drawing

<sup>1</sup> 5:32 AM • Aug 17, 2020 <https://twitter.com/Cshearer41/status/1295292504909254658>

with a line through the centers of the semicircle and quadcircle that is perpendicular to the tangent line. I also added a line from the center of the semicircle to the intersection point of the tangent line. These lines revealed two right triangles whose sides could give me the length of the tangent line. They “appeared” to be congruent, so I tried to prove that. I added the unknown angle  $\alpha$ , and then showed where it, or its complement, appeared elsewhere. Figure 2 showed the two adjacent right triangles which were congruent. It then “looked like”  $\alpha$  divided the semicircle into three equal angles, namely,  $60^\circ$ . From that I would be able to deduce the length of the tangent line.

I spent a day trying to prove geometrically that  $\alpha = 60^\circ$ . Eventually I added the unknown lengths  $x$ ,  $y$  to Figure 2 and could see that if  $\alpha = 60^\circ$ , then the hypotenuse 13 would be twice the short leg of the right triangle, that is, I would get  $5 + y = 13/2$ . Equating the sides of the square and also using the Pythagorean theorem yields two equations,

$$10 + y = 8 + x$$

$$(5 + y)^2 + (8 + x)^2 = 13^2$$

Thus

$$(5 + y)^2 + (10 + y)^2 = 13^2$$

or

$$y^2 + 15y - 44 = 0.$$

So

$$y = (-15 + \sqrt{313})/2 \approx 2.7/2$$

and

$$y + 5 \approx 12.7/2$$

which is *almost*  $13/2$ , but not quite! So  $\alpha \neq 60^\circ$ , but it is close enough to fool Viseo. The moral of this story is that whatever Viseo might suggest, it still must be proved.

So back to the drawing board. I started playing with circles with diameter 13 and noticed that the tangent line seemed to be 13 long. A quick look at the diagram yielded the answer, which is shown in Figure 3. The two yellow right triangles are congruent (they have the same angles and a side in common—the edge of the square, which means the vertical right triangle can be rotated  $90^\circ$  to coincide with the horizontal yellow right triangle), and so have equal hypotenuses. Therefore the length of the tangent line is **13**. Simple when you are not going down a seductive blind alley.

## Solution to Problem #2<sup>2</sup>

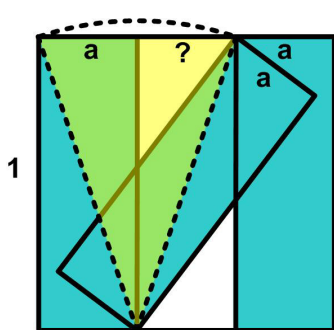


Figure 4

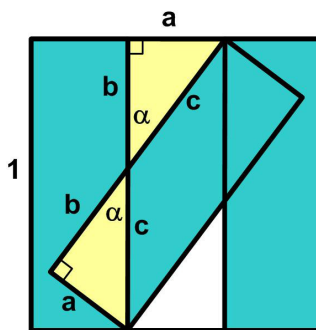


Figure 5

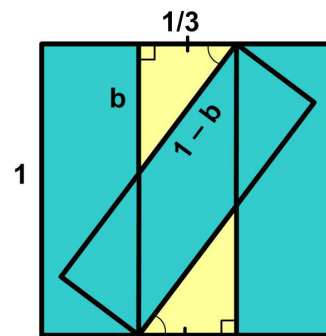


Figure 6

Since we are interested in the ratio of areas, we can assume the square has unit area and then find the fractional area of the overlapping rectangles. First, we have to show the white column is the same width as the colored rectangles. Let the width of the rectangles be  $a$ . Notice that the angled rectangle can be obtained from the left-most rectangle by pivoting it about its lower right corner until its upper left corner coincides with the upper left corner of the right-most rectangle (Figure 4). Therefore the

<sup>2</sup> 5:58 AM • Jul 23, 2020 <https://twitter.com/Cshearer41/status/1286239151940161538>

diagonal of the rotated rectangle coincides with the diagonal of the “white-column” rectangle, making a white right triangle with a side and hypotenuse equal to those of the left-hand right triangle. Thus, the two right triangles are congruent and the width of the white-column rectangle is also  $a$ , and so equal to  $1/3$ .

Now notice in Figure 5 that the two yellow right triangles are congruent since they have two, and thus three angles in common, as well as one side ( $a$ ). Similarly, in Figure 6 the two yellow right triangles are also congruent. From the Pythagorean Theorem we have

$$b^2 + 1/9 = (1 - b)^2 = 1 - 2b + b^2 \Rightarrow 2b = 8/9 \Rightarrow b = 4/9$$

Therefore, the area of the region covered by the colored rectangles is the area of the square minus the areas of the two yellow triangles, that is,

$$\text{Area} = 1 - (1/3)(4/9) = 23/27$$

### Solution to Problem #3<sup>3</sup>

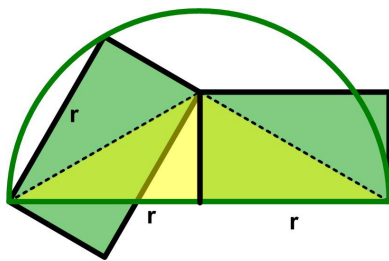


Figure 7

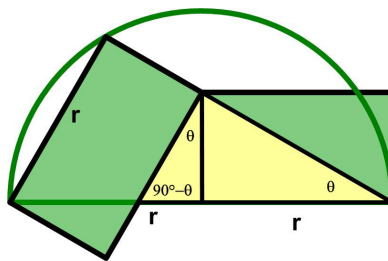


Figure 8

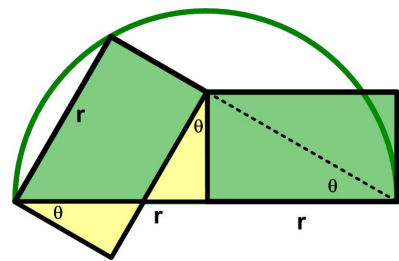


Figure 9

First, we see from Figure 7 that the left edge of the right rectangle is a perpendicular bisector of the diameter of the semicircle (since two sides of a pair of right triangles are equal, the triangles are congruent, thus making third sides equal, namely, the radii of the semicircle).

We are done if we can show the upper edge of the left rectangle extended down to the right coincides with the diagonal of the right rectangle. Let  $\theta$  be the angle shown in the right rectangle (Figure 8). Then  $\theta$  is also the angle in the left yellow triangle. Finally, the two yellow right triangles in Figure 9 are congruent since they are similar with a side in common (the short edge of the rectangle). Therefore the lower edge of the left rectangle and hence the upper edge of the left rectangle is parallel to the diagonal of the right rectangle. And since they have a point in common, the lines coincide.

Therefore, we have a (yellow) right triangle with hypotenuse the diameter of the semicircle (Figure 10), which is twice the length of the shorter leg. Therefore the angle  $\theta$  is  $30^\circ$  and so the desired angle is  $150^\circ$ .

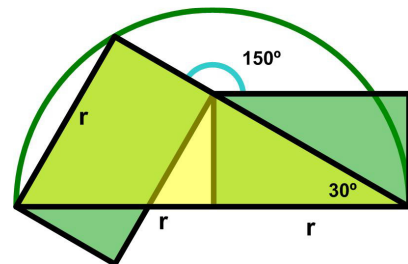


Figure 10

### Solution to Problem #4<sup>4</sup>

Just like in Problem #3 I will choose the unspecified sizes of the half-squares to be equal (Figure 11). Since the circumscribed circle has radius 1, the edge of the resulting full square is  $\sqrt{2}$ , and so the area is 2. That should be the area, then, for any configuration of half-squares.

<sup>3</sup> 3:43 AM • Jul 29, 2020 <https://twitter.com/Cshearer41/status/1288379722586521600>

<sup>4</sup> 3:30 AM • Jul 25, 2020 <https://twitter.com/Cshearer41/status/1286926863344992258>

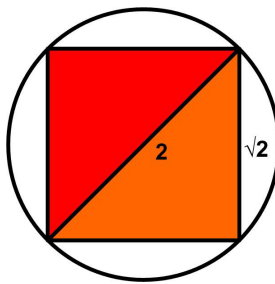


Figure 11

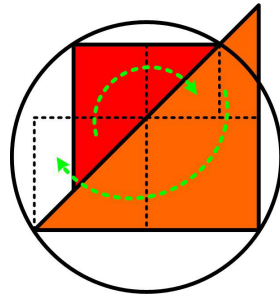


Figure 12

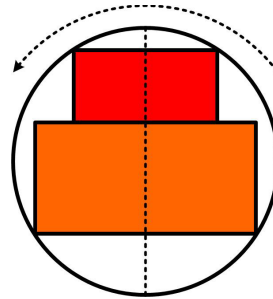


Figure 13

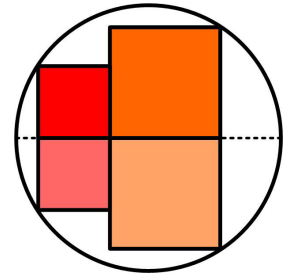


Figure 14

Figure 12 shows the general situation. I have bisected the edges of the half-squares as shown. Then by rearranging the resulting congruent triangles in the half-squares as indicated, we end up with two rectangles composed of two squares each (Figure 13), that is, the diameter of the circle cuts the rectangles into pairs of equal squares. Now rotate the figure counter clockwise  $90^\circ$  to obtain Figure 14. Then we have two semicircles containing the same two squares as given in Presh Talwalkar's Sum of Squares Puzzle.<sup>5</sup> In that puzzle the sides of the two squares were  $x$  and  $y$ , and the sum of their areas was  $x^2 + y^2 = r^2$ , where  $r = 10$ . In our puzzle, the radius  $r = 1$ , so the sum of the areas of the squares in the semicircle is 1. But we have two semicircles, so the total area is 2, which is just what we predicted.

We followed the time-tested math principle of reducing the problem to one we already know how to solve.

I checked Catriona's site and I did not see solutions like mine.

## Solution to Problem #5<sup>6</sup>

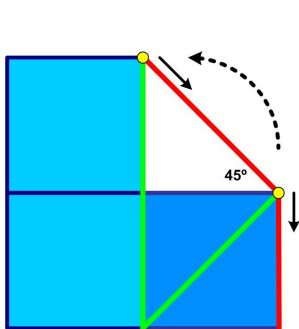


Figure 15

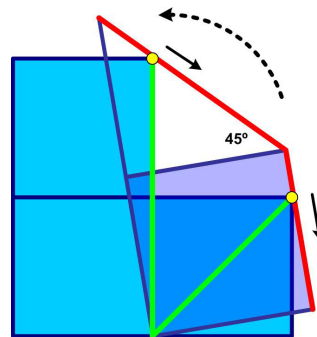


Figure 16

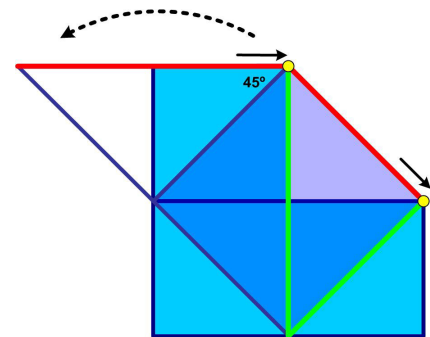


Figure 17

So again the position and size of the dark blue square was not specified, so I chose the simplest, which was equal to one of the light blue squares as shown in Figure 15. That meant the desired angle was  $45^\circ$ . Again I will show the position and size of the dark blue square does not matter.

Figure 16 and Figure 17 show the interesting result that as the dark blue square rotates and enlarges so that its edge continues to touch the corner of the lower right light blue square, the upper red line keeps passing through the upper right corner of the upper left blue square.

This reminded me of Problem #4 in Catriona Agg's previous collection, Geometric Puzzle Munificence,<sup>7</sup> and the Curve Making Puzzle.<sup>8</sup> Figure 18 makes this clear by changing the point of

<sup>5</sup> <http://josmfs.net/2020/09/05/sum-of-squares-puzzle/>

<sup>6</sup> 4:21 AM • Aug 14, 2020 <https://twitter.com/Cshearer41/status/1294187497229033472>

<sup>7</sup> <https://josmfs.net/2020/11/28/geometric-puzzle-marvels/>

<sup>8</sup> <http://josmfs.net/2020/11/07/curve-making-puzzle/>

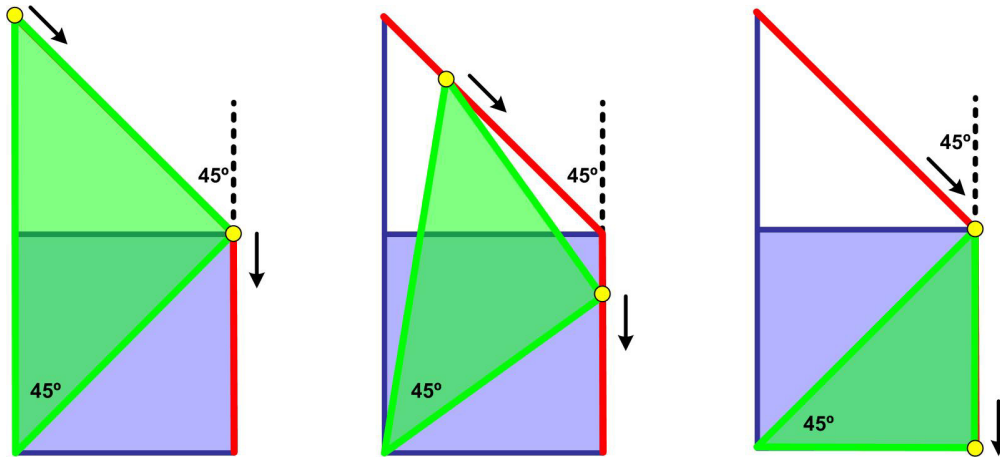


Figure 18

view from having the dark blue square move as shown in Figure 15 - Figure 17 to fixing its position and letting the light green lines slide along the red line. The green lines define a set of similar triangles. As the endpoint of the right-hand-leg of the triangle slides down the vertical edge of the blue square, the endpoint of the left-hand-leg of the triangle slides along a similar curve, namely, a straight line. Since the left-hand-leg is tilted  $45^\circ$  with respect to the right-hand-leg, the line it traces will be tilted by  $45^\circ$  with respect to the line traced by the right-hand-leg.

When I checked the solutions at Catriona's Twitter site, no one used this argument. Most solutions claimed the upper right vertex of the dark square traced out the arc of a circle, or equivalently claimed that the relevant vertices were cyclic, that is, lay along a circle. The justification for this seemed a bit obscure to me and relied on properties of cyclic points that I was not familiar with.

## Solution to Problem #6<sup>9</sup>

Aside from my detour in Problem #1, this problem turned out to be the most challenging.

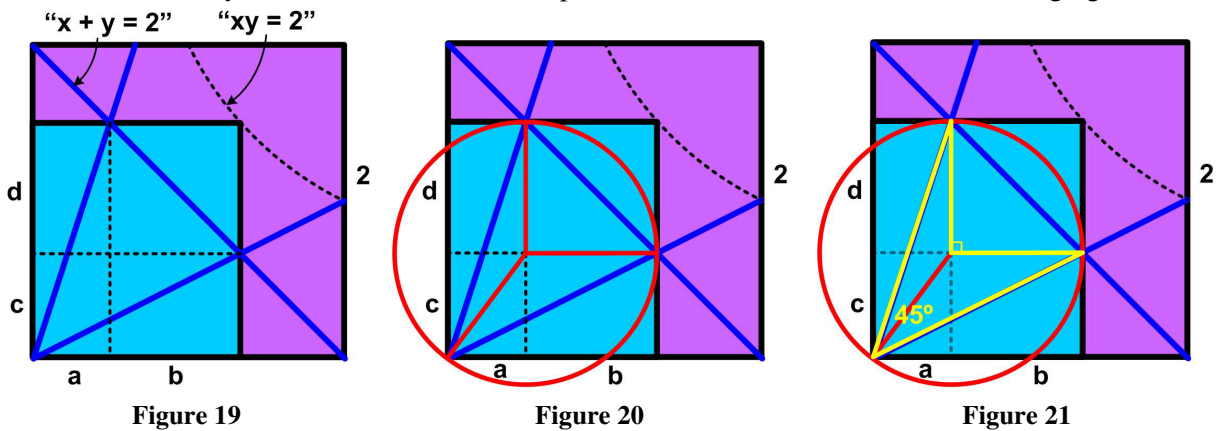


Figure 19

Figure 20

Figure 21

For convenience we assume the square has edges of length 2. Then its area will be 4 and that of the blue rectangle half that, or 2. Figure 19 shows the original problem statement with some (dashed) lines added, along with some annotations. The dashed curve in the upper right corner represents a hyperbola of the form  $xy = 2$ . So any rectangle with area 2 will have its upper right corner lying on the hyperbola. Secondly, the diagonal blue line passing from one corner of the square down to the

<sup>9</sup> 7:36 AM • Jul 26, 2020 <https://twitter.com/Cshearer41/status/1287350989536866305>

other has an equation of the form  $x + y = 2$ . These statements yield the following equations in the designated variables.

$$(a + b) + c = 2$$

$$a + (c + d) = 2$$

Therefore,  $b - d = 0$ , or  $b = d$ , which means the upper right region of the blue rectangle is a square with sides  $b$ . Furthermore,  $a + c = 2 - b$ . From the area constraint we have

$$2 = (a + b)(c + d) = ac + b(a + c) + b^2 = ac + b(2 - b) + b^2 = ac + 2b$$

So 
$$ac = 2 - 2b \tag{*}$$

Now we draw a circle of radius  $b$ , centered as shown in Figure 20, and passing through the two points of intersection of the blue lines with the blue diagonal. The figure shows the circle also passes through the lower left corner of the square, but we must prove that. This is equivalent of showing  $a^2 + c^2 = b^2$ . We need expressions involving the squares of these variables.

$$a + c = 2 - b \Rightarrow (a + c)^2 = (2 - b)^2 \Rightarrow a^2 + 2ac + c^2 = 4 - 4b + b^2$$

From this result and equation (\*) we get

$$a^2 + (4 - 4b) + c^2 = 4 - 4b + b^2 \Rightarrow a^2 + c^2 = b^2$$

which is what we wanted to show.

Therefore Figure 21 obtains and we see that the unknown angle we are searching for is the inscribed angle of the red circle with central angle of  $90^\circ$ . Hence our desired angle is half of that, or  $45^\circ$ .

**Comment.** Finding the critical red circle was serendipitous—just playing around in Viseo. I looked at a related circle and was quite surprised by its properties (Figure 22). The new circle is the old circle reflected about the diagonal of the square. Its center will be the diagonally opposite corner of the small square region in the blue rectangle. Since the original circle passed through a vertex of the large square, so will the reflected version. But what is really startling is that the intersections of this new circle with the large square are also the points of intersection of the blue lines with the edges of the square. I haven't bothered to prove it, but it looks to be true.

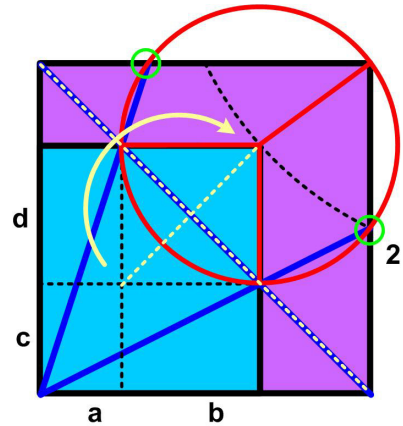


Figure 22

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