Train Wreck Puzzle

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I thought it might be interesting to explore the mathematics of a common problem with a store-bought HO model train set that contains a collection of straight track segments and fixed-radius curved track segments that form a simple oval (Figure 1). Invariably an initial run of the train has it careening off the track when the train first meets the curved segment after running along the straight track segments.

Why is that? Well of course the train is going too fast. But even if it slows down enough not to fall off the curve, it still jerks unstably and may derail when it first reaches the beginning of the curve. What is going on?

Math

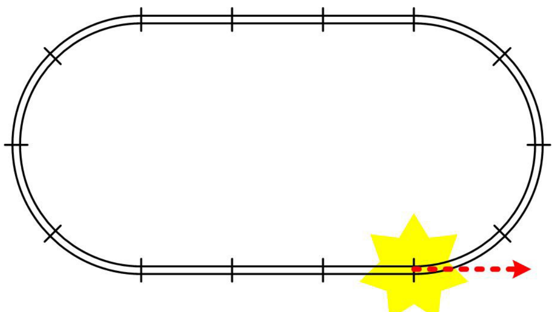

Figure 1 Model Train Layout with Problem

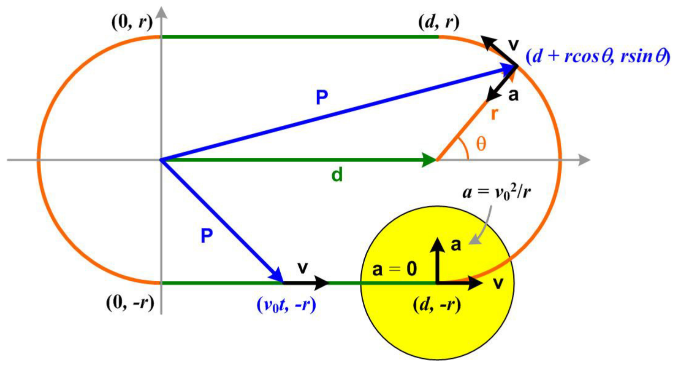

Figure 2 Parameterized Layout

We will first create a mathematical model of the problem. The track layout is determined by the distance d of the straight tracks and radius r of the curved tracks (Figure 2). We have the distance s traveled around the curves is given by $s = r\theta$. Since the train is moving at constant speed v_0 along the track, the speed along the curves is given by

$$v_0 = ds/dt = r\, d\theta/dt = r\ \theta'(t)$$

Therefore, the angular velocity is also constant and given by

$$d\theta/dt = \theta'(t) = v_0/r$$

And so the general scheme for the time evolution of θ is

$$\theta(t) = (v_0/r)\ t$$

Given that the track consists of straight and curved segments, we define times marking the transitions between these segments, starting with the bottom straight track, as t_0, t_1, t_2, t_3, t_4, given by

$$\begin{array}{lll} & & \boldsymbol{t_0 = 0} \\ d = v_0\, t_1 & \Rightarrow & \boldsymbol{t_1 = d/v_0} \\ \pi = (v_0/r)(t_2 - t_1) & \Rightarrow & \boldsymbol{t_2 = \pi r/v_0 + t_1} \\ d = v_0\,(t_3 - t_2) & \Rightarrow & \boldsymbol{t_3 = d/v_0 + t_2} \\ \pi = (v_0/r)(t_2 - t_1) & \Rightarrow & \boldsymbol{t_4 = \pi r/v_0 + t_3} \end{array}$$

The following table shows the parameterization of the position, velocity, and acceleration vectors shown in Figure 2 corresponding to the colored time intervals for the straight and curved sections of the track layout.

Table 1 Components of Position, Velocity, and Acceleration Vectors

		$t_0 = 0$	t_1	t_2	t_3 t_4
			$\theta(t) = (v_0/r)(t - t_1) - \pi/2$		$\theta(t) = (v_0/r)(t - t_3) + \pi/2$
$\mathbf{P}(t)$	$x(t)$	$v_0\, t$	$d + r \cos \theta(t)$	$d - v_0\,(t - t_2)$	$r \cos \theta(t)$
	$y(t)$	$-r$	$r \sin \theta(t)$	r	$r \sin \theta(t)$
$\mathbf{v}(t)$	$x'(t)$	v_0	$-r \sin \theta(t)\ \theta'(t) = -v_0 \sin \theta(t)$	$-v_0$	$-r \sin \theta(t)\ \theta'(t) = -v_0 \sin \theta(t)$
	$y'(t)$	0	$r \cos \theta(t)\ \theta'(t) = v_0 \cos \theta(t)$	0	$r \cos \theta(t)\ \theta'(t) = v_0 \cos \theta(t)$
$\mathbf{a}(t)$	$x''(t)$	0	$- v_0 \cos \theta(t)\ \theta'(t) = -v_0^2/r \cos \theta(t)$	0	$- v_0 \cos \theta(t)\ \theta'(t) = -v_0^2/r \cos \theta(t)$
	$y''(t)$	0	$- v_0 \sin \theta(t)\ \theta'(t) = - v_0^2/r \sin \theta(t)$	0	$- v_0 \sin \theta(t)\ \theta'(t) = - v_0^2/r \sin \theta(t)$

From a vector point of view we have unit vectors in the direction of the vector $\mathbf{r}$ and perpendicular to $\mathbf{r}$ given by

$$\mathbf{u}(t) = \cos\ \theta(t)\ \mathbf{i} + \sin\ \theta(t)\ \mathbf{j}$$

$$\mathbf{u}^{\perp}(t) = -\sin\ \theta(t)\ \mathbf{i} + \cos\ \theta(t)\ \mathbf{j}$$

and so can write the position vector $\mathbf{P}$ for the right-hand curve as

$$\mathbf{P}(t) = \mathbf{d} + \mathbf{r}(t) \quad \text{where} \quad \mathbf{r}(t) = r\,\mathbf{u}(t)$$

From Table 1 we have

$$\mathbf{P'}(t) = \mathbf{v}(t) = v_0\, \mathbf{u}^{\perp}(t)$$

$$\mathbf{P''}(t) = \mathbf{a}(t) = - v_0^2/r\, \mathbf{u}(t)$$

We are interested in what happens at time t_1 where the straight track transitions to the curved track.

Continuity of $\mathbf{P}$. $\mathbf{P}(t)$ is continuous on the closed interval $[t_0, t_1]$ and continuous on the closed interval $[t_1, t_2]$ because $\theta(t)$, $\cos\ \theta(t)$, and $\sin\ \theta(t)$ are. Now $\mathbf{P}(t_1) = d\,\mathbf{i} - r\,\mathbf{j}$ and as $t \to t_1^-$, $\mathbf{P}(t) \to d\,\mathbf{i} - r\,\mathbf{j}$, and as $t \to t_1^+$, $\mathbf{P}(t) \to d\,\mathbf{i} - r\,\mathbf{j}$ also, since $\theta(t) \to -\pi/2$ means $\cos\ \theta(t) \to 0$ and $\sin\ \theta(t) \to -1$. So $\mathbf{P}(t)$ is continuous at t_1 and thus on the joined interval $[t_0, t_2]$. (See Figure 2.)

Continuity of $\mathbf{v}$. Moreover $\mathbf{v}(t)$ is also continuous on the closed intervals $[t_0, t_1]$ and $[t_1, t_2]$ because $\theta(t)$, $\cos\ \theta(t)$, and $\sin\ \theta(t)$ are. Now $\mathbf{v}(t_1) = v_0\,\mathbf{i}$ and as $t \to t_1^-$, $\mathbf{v}(t) \to v_0\,\mathbf{i}$, and as $t \to t_1^+$, $\mathbf{v}(t) \to v_0\,\mathbf{i}$ also, since again $\theta(t) \to -\pi/2$ means $\cos\ \theta(t) \to 0$ and $\sin\ \theta(t) \to -1$. So $\mathbf{v}(t)$ is continuous at t_1 and thus on the joined interval $[t_0, t_2]$.

Discontinuity of $\mathbf{a}$. Now $\mathbf{a}(t)$ is also continuous on the open intervals (t_0, t_1) and (t_1, t_2) omitting t_1 because $\theta(t)$, $\cos\ \theta(t)$, and $\sin\ \theta(t)$ are. But it is *not* continuous at t_1. As $t \to t_1^-$, $\mathbf{a}(t) = \mathbf{0} \to \mathbf{0}$, but as $t \to t_1^+$, $\mathbf{a}(t) \to v_0^2/r\,\mathbf{j}$, since $\theta(t) \to -\pi/2$ means $\cos\ \theta(t) \to 0$ and $\sin\ \theta(t) \to -1$. Therefore,

$$\lim_{t \to t_1^-} \mathbf{a}(t) = \mathbf{0} \neq \frac{v_0^2}{r}\mathbf{j} = \lim_{t \to t_1^+} \mathbf{a}(t),$$

so $\mathbf{a}(t)$ approaches different limits as t approaches t_1 from the left or the right (and we haven't decided on what to define $\mathbf{a}(t_1)$ either).

Physics

Acceleration on curve. What is the effect of this sudden acceleration on the train? Let's consider first the acceleration of the train while it is on the curve. From physics we have $\mathbf{F} = m\mathbf{a}$, that is, every acceleration of an object of mass m is associated with a force. So from Figure 2 we see that the acceleration $\mathbf{a}$ is directed radially toward the center of the semicircular curve. Such a force is called a *centripetal force*. In our case it will be given by

$$\mathbf{F} = -mv_0^2/r\ \mathbf{u}$$

where $\mathbf{u}$ is the unit vector in the direction of the radial vector $\mathbf{r}$ to the position of the train on the curve (m is a little vague at the moment).

To get a feel for what is happening, imagine the railroad car on the curve is actual size and a person is standing on its floor inside (Figure 3). Suppose the person has a mass m and thus weight mg where g is the acceleration of gravity. Then the centripetal force of the curve on the car (red arrow) is felt by the feet of the standing person. However, their head keeps trying to go in a straight line ahead. To see this, imagine another observer standing in the tracks looking at the instantaneous cross-section of the car. That observer will see the passenger's head going straight forward but their feet moving to the left. The passenger in the car, whose reference is only the floor of the car, will see their feet implanted unmoving on the floor but feel their head trying to go to the right (with the same force as the grounded observer sees the feet going to the left). This sensation by the passenger of their head being forced to the right is called the *centrifugal force*, but is not a real force. It is merely the result of a change of reference coordinate system (from a fixed system to a rotating one); no actual force is being applied to the passenger's head. The only real force is the centripetal force moving the passenger's feet in a curve. Nevertheless, if the passenger does not do something, they will fall over. In other words, the passenger needs to have their head subject to the centripetal force as well, such as holding onto the sides of the car.

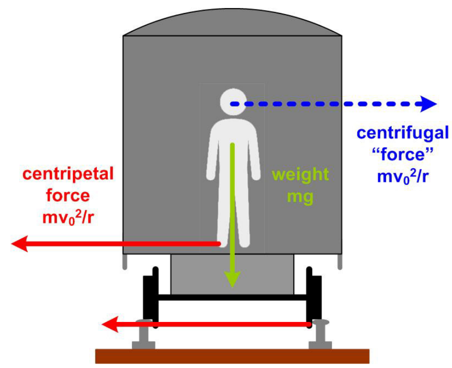

Figure 3

Of course, the whole train car is being subject to centrifugal forces, but the rigid sides transmit the centripetal force throughout (Figure 4). There is the issue that the force is actually only applied to the outer wheels, and so considerations of torques are involved that we will mention briefly. One way to mitigate the torque is to angle the lever arm against the force (torque is perpendicular to the lever arm) and add a little counter torque. This is done by banking the roadbed to allow some of the weight of the car to be added to the centripetal force (Figure 5).

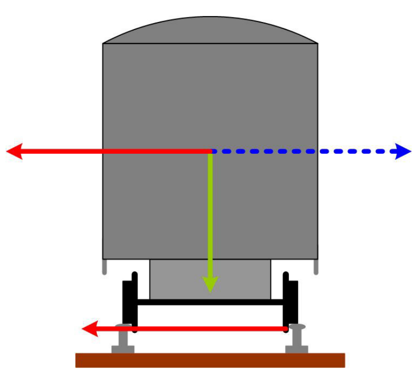

Figure 4

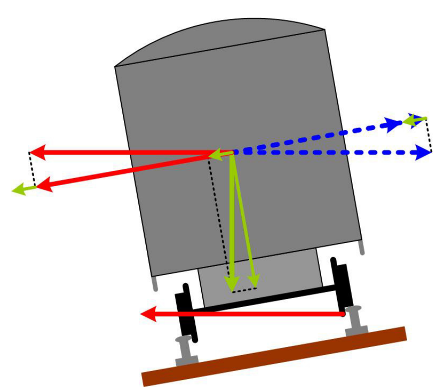

Figure 5

Abrupt change of acceleration. Given that the curved tracks in the train set are fixed, so that we cannot widen the radius of curvature and thus lessen the centrifugal force, our only alternative is to slow down the train—reduce the speed v_0. Actually, this makes a big difference, more than increasing the radius of curvature, since the effect of the speed goes as its square.

Now consider, not the situation on the curve, but what happens when the train transitions from the straight track to the curved track. Because of the discontinuity of the acceleration, we go from no force to a sudden impulse force of mv_0^2/r. In other words, the sudden centrifugal force whacks the train sideways—the acceleration is abruptly accelerating from zero to mv_0^2/r. Given the rigidity of the track geometry in the train set, there is nothing we can do, except again try to keep the speed v_0 small enough so the impulse force is not too great.

Easement. In a "real" model train layout, these problems can be addressed. Besides being able to bank the road bed (which the model railroaders seem to call "superelevation" ([1])) they remove the abrupt change of the radius of curvature of the track from the straight segments to the curved segments. They "smooth" the transition with what they call an "easement."([1], [2]) (In mathematics a *smooth* function is one that is infinitely differentiable (often called "C-infinity" C^∞).[1] Thus it has continuous derivatives of all orders. A curve defined by such a function would not have a discontinuous second derivative leading to a discontinuous acceleration.) The model railroaders approximate such a curve for their easement using an offset technique (Figure 6). A video on model railroad easement and banking construction can be found at ([3]).

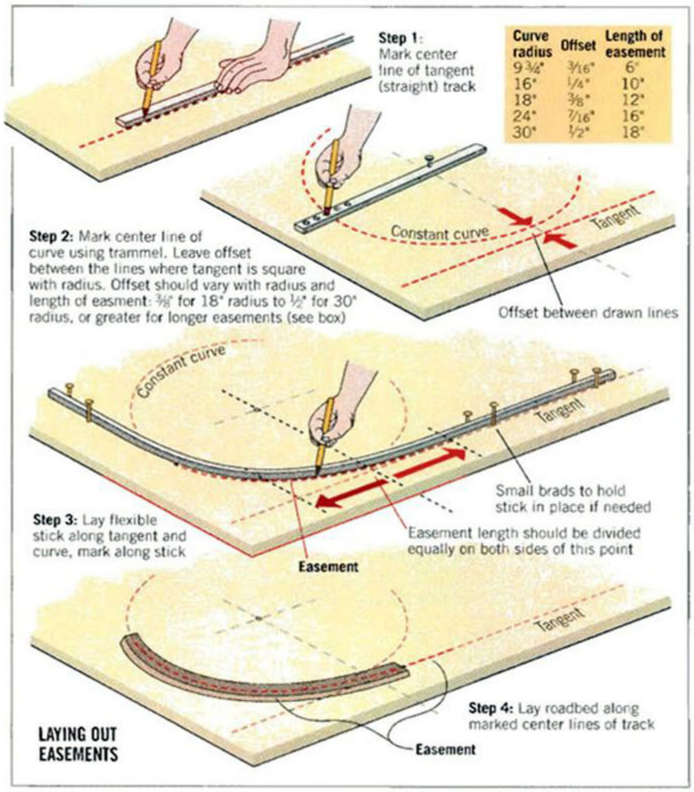

Figure 6 Model Railroad Easement Construction ([2])

References

[1] Griffin, Walter, "Easements", October 16, 2018 (https://www.eldoradosoft.com/easements.htm, retrieved 2/27/2021)

[2] McGuirk, Marty, "Easy Easements For Model Train Track," May 23, 2017 (https://mrr.trains.com/how-to/track-planning-operation/2017/05/easy-easements-for-model-train-track, retrieved 2/27/2021)

[3] Ron's Trains N Things, "Model Railroad Track Laying Tips Curves Easements and Superelevation", 22 August 2017 (https://www.youtube.com/watch?v=Nu9qlIckU7Q)

[1] https://en.wikipedia.org/wiki/Smoothness