Nested Polygons Puzzle

31 January 2021

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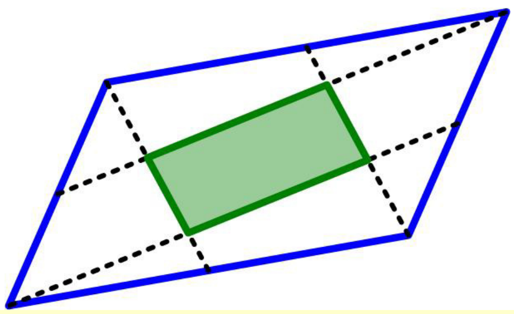
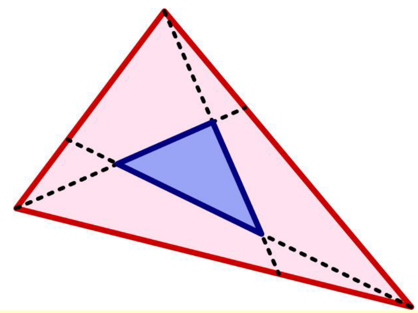




Start with any parallelogram. Mark the midpoint of each side. Join these midpoints to the vertex two places clockwise around the parallelogram. What fraction of the original parallelogram is the new quadrilateral?

If each vertex of a triangle is connected to a point one third of the way along the opposite side, you create a new triangle. What is the ratio of the area of the smaller triangle to the area of the larger one? Can you prove it?

These two interesting problems were posed on MEI's MathsMonday site on 3 February 2020[1] and 2 March 2020,[2] respectively. MEI and readers posted various approaches, but I used a method suggested by another problem whose origin I no longer recall.

My Solution

The similar solutions are shown in Figure 1 and Figure 2. They involve adding a grid of parallel lines to the original lines and using the spacing dictated by the vertices of both the nested polygon and enclosing polygon. From the premises of the problems these parallel lines will cut the edges of the enclosing polygons in an integral number of segments, making the dotted figures all congruent to each other and to the corresponding nested polygon. Thus $A_G = A_B/5$ and $A_B = A_R/7$.

Actually there is some hand-waving in this statement that needs a bit more detailed argument.

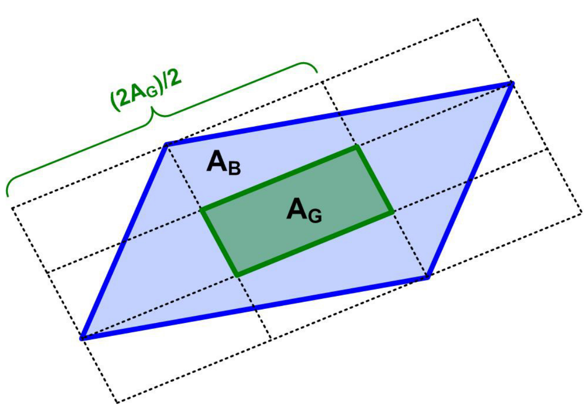


Figure 1. $A_B = 9A_G - 4(2A_G/2) = 5\ A_G$

$\Rightarrow$ $\mathbf{A_G = A_B/5}$

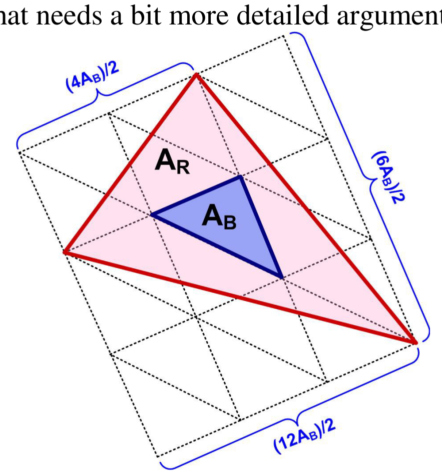


Figure 2. $A_R = 18A_B - (4A_B)/2 - (6A_B)/2 - (12A_B)/2 = 7\ A_B$

$\Rightarrow$ $\mathbf{A_B = A_R/7}$

[1] https://twitter.com/MEIMaths/status/1224271310496206848
[2] https://twitter.com/MEIMaths/status/1234418167868469251

Quadrilateral Grid Justification

First, we need to show the nested quadrilateral is a parallelogram, that is, the dotted lines through its sides are parallel. Then we need to show the added dotted lines through the vertices of the blue parallelogram, parallel to the original dotted lines, are the same distance apart as the originals, thus making all the quadrilaterals in the grid congruent.

Green Quadrilateral is Parallelogram. Figure 3 shows a red quadrilateral composed of two congruent triangles, since the downward slanting lines go through the midpoints of the parallel opposite sides. Therefore the other sides are parallel and of the same length, making the red quadrilateral a parallelogram. Hence, two of the green quadrilateral sides are parallel. Using the same argument on the upward slanting lines proves the other pair of sides of the green quadrilateral are parallel, so that the quadrilateral is a parallelogram.

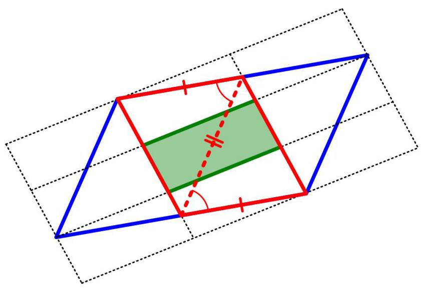


Figure 3

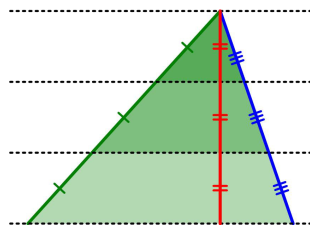


Figure 4

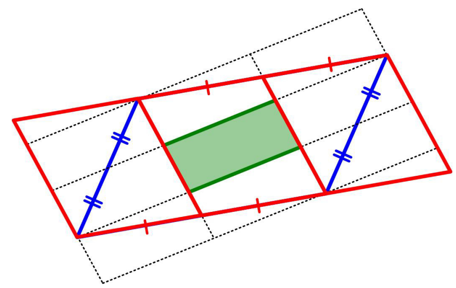


Figure 5

Claim. *If a set of parallel lines cuts off equal intervals along a line transverse to the parallel lines, then the parallel lines are equally-spaced and will cut off equal intervals on any other transverse line.*

Proof. Figure 4 shows a (green) transverse line crossing a set of parallel lines in equal intervals. Considering any other transverse line (e.g. the blue line) we can define a set of nested, similar (green) triangles with the two transverse lines as their sides. The green sides being in ratios of 1:2:3: … :n, depending on the number of parallel lines, implies that the blue sides will be in the same ratios of 1:2:3: … :n. Therefore, the blue line will consist of equal intervals cut by the parallel lines. In particular the altitudes of the triangles will be in the same ratios and so divide the perpendicular (red) line into equal intervals, which implies the parallel lines are equally-space.

Dotted Grid Lines are Equally-Spaced. Using the Claim, Figure 5 shows that the downward slanting parallel dotted lines cut off equal intervals along the red lines and so are equally-spaced. Similarly, the upward slanting parallel dotted lines cut off equal intervals along the blue lines and so are equally-spaced.

Therefore the grid is made up of 9 replicas of the green quadrilateral.

Triangle Grid Justification

The justification for the triangle grid is more complicated.

First, layout a grid of parallel lines determined by two (blue) of the original dotted lines (Figure 6) and two added (dotted green) lines that cut the two (red) sides into thirds. From the Claim this means the parallel grid lines are equally-spaced and also cut the sides of the grid into thirds.

We are done if we can show the diagonal dotted lines cut the third side of the red triangle into thirds. Figure 7 shows we have added another quadrilateral to the grid with its corresponding diagonal. This shows the diagonal dotted lines cut the slanted green line in equal intervals (thirds) and thus by the Claim cut the third side of the red triangle into thirds. Hence, the nested blue triangle is congruent to all 18 of the triangles in the grid of Figure 2.

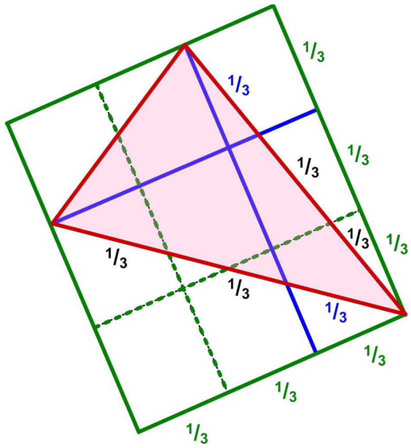


Figure 6

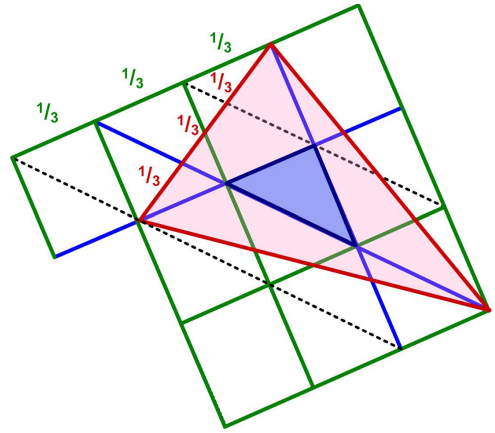


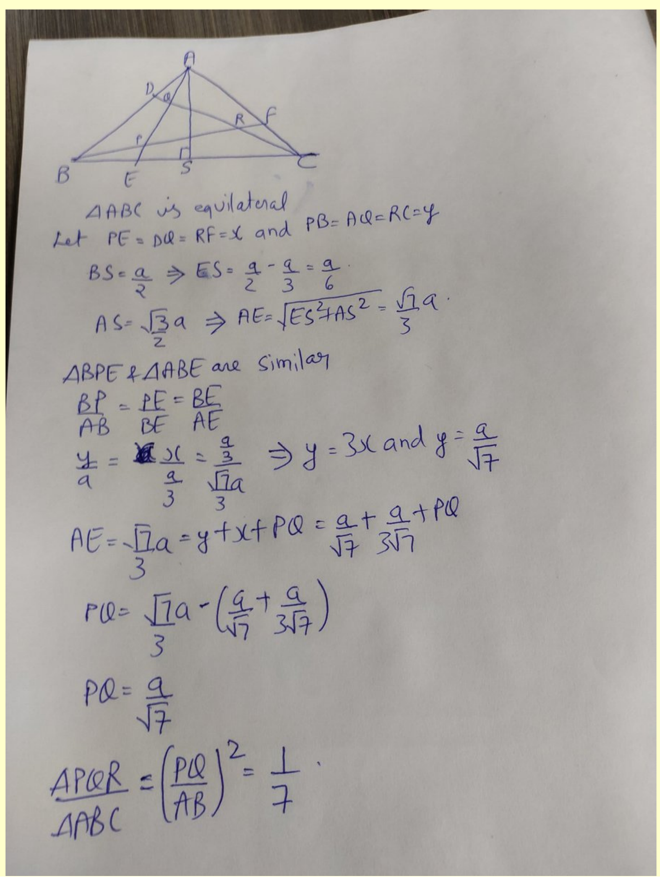
Figure 7

MathsMonday Solutions

The following are two solutions for the nested triangle problem.

Expert_ Says (6:24 AM · Mar 2, 2020):[3]

Here is the proof for equilateral triangle and it follows for every triangle [JOS: why?].



ΔABC is equilateral

Let $PE = DQ = RF = x$ and $PB = AQ = RC = y$

$BS = \frac{a}{2} \Rightarrow ES = \frac{a}{2} - \frac{a}{3} = \frac{a}{6}$

$AS = \frac{\sqrt{3}}{2}a \Rightarrow AE = \sqrt{ES^2 + AS^2} = \frac{\sqrt{7}}{3}a$

ΔBPE & ΔABE are similar

$\frac{BP}{AB} = \frac{PE}{BE} = \frac{BE}{AE}$

$\frac{y}{a} = \frac{x}{\frac{a}{3}} = \frac{\frac{a}{3}}{\frac{\sqrt{7}a}{3}} \Rightarrow y = 3x$ and $y = \frac{a}{\sqrt{7}}$

$AE = \frac{\sqrt{7}}{3}a = y + x + PQ = \frac{a}{\sqrt{7}} + \frac{a}{3\sqrt{7}} + PQ$

$PQ = \frac{\sqrt{7}}{3}a - \left(\frac{a}{\sqrt{7}} + \frac{a}{3\sqrt{7}}\right)$

$PQ = \frac{a}{\sqrt{7}}$

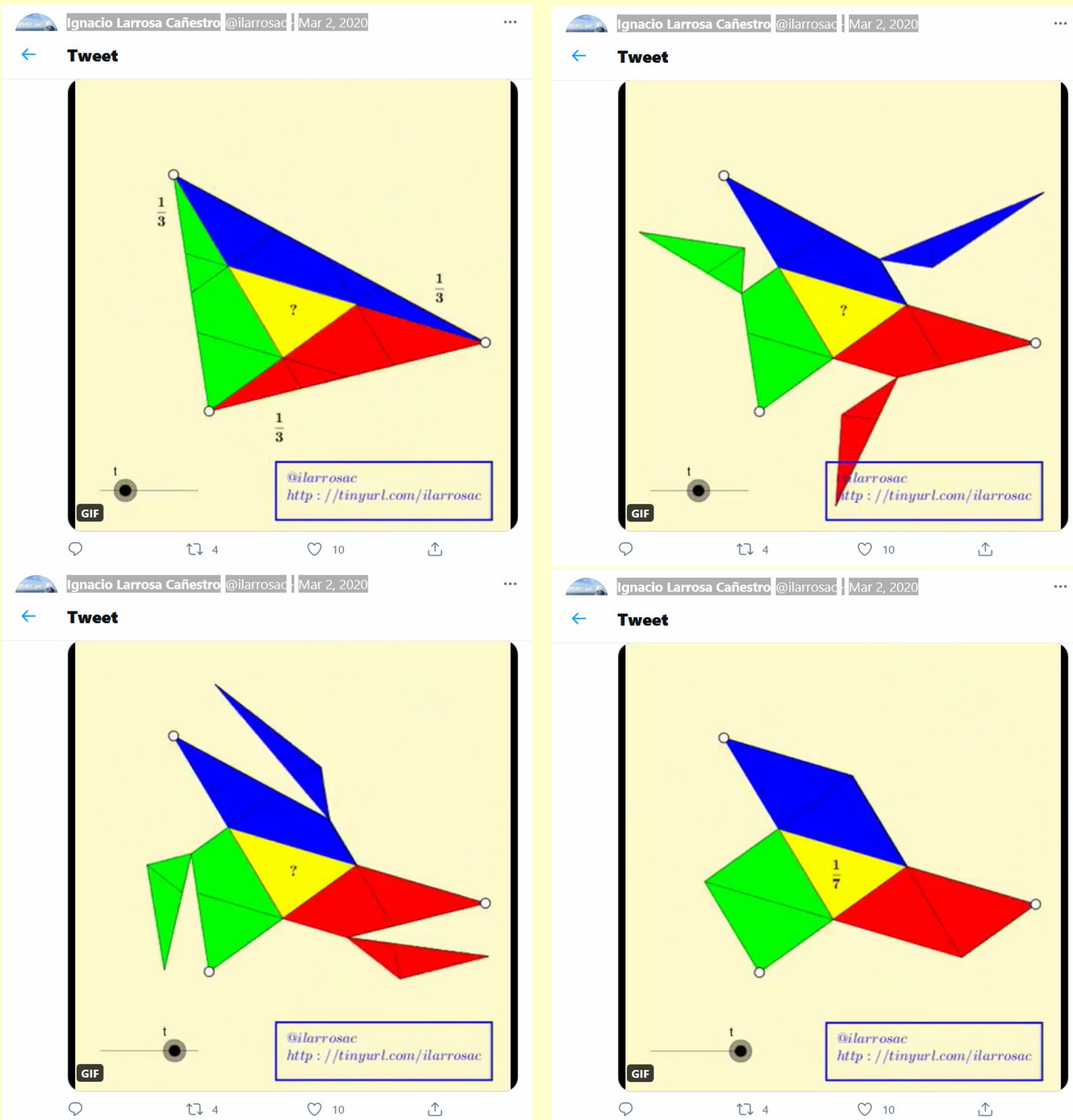
$\frac{\Delta PQR}{\Delta ABC} = \left(\frac{PQ}{AB}\right)^2 = \frac{1}{7}$.

[3] https://twitter.com/Expert_Says/status/1234439484759998465 "Professor, Mentor, Storyteller, Wiseass, Librocubicularist"

The next solution is an animated GIF from which I captured some strategic snapshots.

Ignacio Larrosa Cañestro (2:56 PM · Mar 2, 2020)[4]

Without words



As I have found with Visio, a graphic "proof" is not really a proof unless accompanied by some mathematical justification. This certainly makes the result 1/7 look plausible, however.

[4] https://twitter.com/ilarrosac/status/1234568193940901888

MEI Nested Quadrilateral Solution [5]

The resulting quadrilateral is 1/5 of the area of the original parallelogram.

Triangle ADE is similar to triangle ABC and AD is twice AB [Figure 8]. Therefore AC is the same length as CE. As DE is the same length as EG, by a similar argument, and CF is the same length as EG, then BC is half the length of CF.

Rotating the triangle ABC around the point B by a half-turn, and doing similar for the other three small triangles gives [Figure 9].

This is a cross with the same area as the original parallelogram made up of five copies of the smaller quadrilateral.

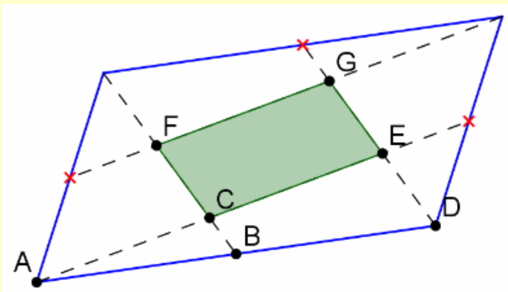


Figure 8

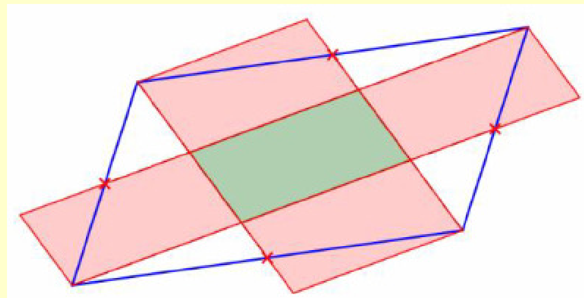


Figure 9

The MEI solution offers the mathematical justification for the geometric transformations that Cañestro omitted from the similar geometric transformations in his GIF.

[5] https://mei.org.uk/files/miotm_solutions/sep-16-soln.pdf