Three Runners Puzzle

27 December 2019

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Here is another problem from Presh Talwalkar which he says is adapted from India's Civil Services Exam.[1]

There are three runners X, Y, and Z. Each runs with a different uniform speed in a 1000 meters race. If X gives Y a start of 50 meters, they will finish the race at the same time. If X gives Z a start of 69 meters, they will finish the race at the same time. Suppose Y and Z are in a [1000 meter] race. How much of a start should Y give to Z so they would finish the race at the same time?

Even though Talwalkar's original graphic showed all the runners in a 1000 meter race, it was not immediately clear to me from the wording that the race between Y and Z was also 1000 meters. But that was the case, so I made it explicit.

My Solution

Let v_X, v_Y, and v_Z be the speeds of X, Y, and Z respectively, and let t_0 be the time X takes to run the 1000 meter race. Then, given their respective head starts, we have the three equations

$$1000 = v_X\, t_0$$

$$1000 = 50 + v_Y\, t_0$$

$$1000 = 69 + v_Z\, t_0$$

This means the speeds of each runner are

$$v_X = 1000/t_0, \quad v_Y = 950/t_0, \quad \text{and} \quad v_Z = 931/t_0$$

Now consider the 1000 meter race between Y and Z where Y gives Z the head start of d meters. Then

$$v_Y\, t_1 = d + v_Z\, t_1 \quad \Rightarrow \quad d = (v_Y - v_Z)\, t_1$$

where t_1 is the time for Y and Z to run 1000 meters given Z's head start. But

$$v_Y - v_Z = 19/t_0 \quad \Rightarrow \quad d = (v_Y - v_Z)\, t_1 = 19\, t_1/t_0$$

The relation between t_0 and t_1 is given by the respective times for running 1000 meters, namely

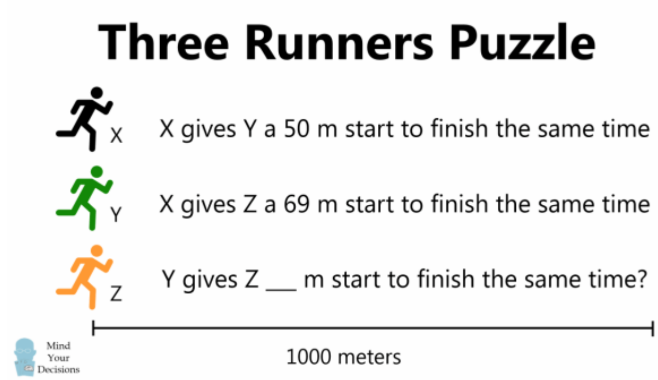
$$v_X\, t_0 = 1000 = v_Y\, t_1 \quad \Rightarrow \quad t_1 / t_0 = v_X / v_Y = 1000 / 950$$

Therefore,

$$d = 19 \cdot 1000 / 950 = 20 \text{ meters}$$

[1] https://mindyourdecisions.com/blog/2019/12/26/three-runners-puzzle-from-indias-civil-services-exam

Talwalkar's Solution



You might guess the answer is 69 – 50 = 19. That is close but it's not correct! The reason is Y needs to give Z a 19 m start if they were running a 950 m race. But we need the answer for a 1000 m race. We can use the formula distance = (rate)(time), which is equivalently time = (distance)/(rate). Use lower-case letters to indicate each runner's speed (rate of running).

X and Y run the same time when X runs 1000 meters and Y runs 1000 – 50 = 950 meters. Thus we have the equation:

$$1000/x = 950/y$$

$$y = 0.95x$$

Y is 95% the speed of X.

X and Z run the same time when X runs 1000 meters and Z runs 1000 – 69 = 931 meters. Thus we have the equation:

$$1000/x = 931/z$$

$$Z = 0.931x$$

Z is 93.1% the speed of X.

Both equations equal $1000/x$, so we can equate the right hand sides.

$$950/y = 931/z$$

Now multiply both sides by 1000/950 to solve for the time it takes y to run 1000 meters. This gives:

$$1000/y = (931/z)(1000/950)$$

$$1000/y = 980/z$$

$$z = .98y$$

This means Y completes 1000 meters in the time Z would complete 980 meters (or Z is 98% the speed of Y). Thus Y needs to give Z a 20 meter start.

Comment. The graphic above may be misleading. You need to be careful about the phrase "same time". "Same time" in the first two races is the time X took to run a 1000 yard race, which I called t_0 above. "Same time" in the third race in the graphic is the time Y takes to run the 1000 yard race, which I called t_1. So $t_0 \neq t_1$ in general.