Twin Intersection Puzzle

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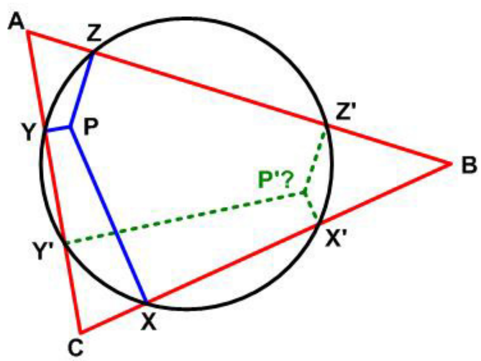

This is an interesting problem from the 1977 Canadian Math Society's magazine, *Crux Mathematicorum* ([1]).

206. [1977: 10] *Proposed by Dan Pedoe, University of Minnesota.*

A circle intersects the sides BC, CA and AB of a triangle ABC in the pairs of points X, X', Y, Y' and Z, Z' respectively. If the perpendiculars at X, Y and Z to the respective sides BC, CA and AB are concurrent at a point P, prove that the respective perpendiculars at X', Y' and Z' to the sides BC, CA and AB are concurrent at a point P'.

My Solution

First, construct the perpendicular bisectors of the line segments XX', YY' and ZZ' (Figure 1). Then these three lines all intersect at the center O of the circle.

Next, draw a line from P through the center O until it intersects the perpendicular at X' at point P_X (Figure 2). The perpendiculars at X, X', and through O are all parallel and equally-spaced, since the perpendicular though O is also a bisector of XX'. Therefore the two blue right triangles, which are similar, are also congruent. This means O bisects the line PP_X.

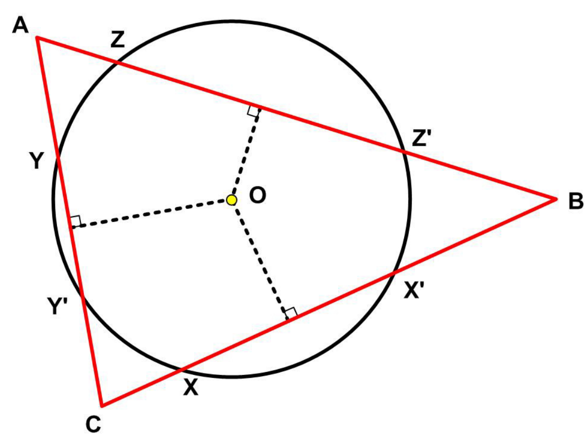

Figure 1

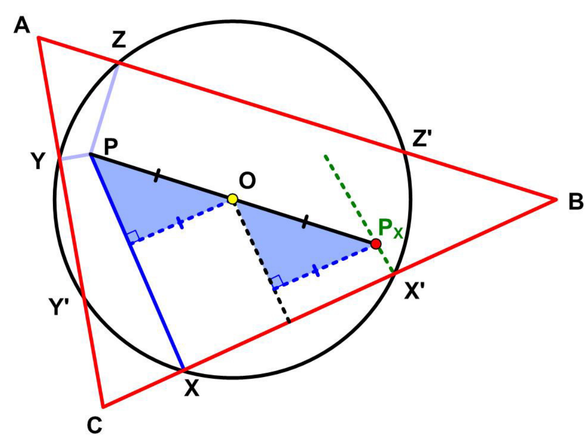

Figure 2

Similarly, draw a line from P through the center O until it intersects the perpendicular at Y' at point P_Y (Figure 3). The perpendiculars at Y, Y', and through O are all parallel and equally-spaced as well. So we have two more congruent blue right triangles that imply that O bisects the line PP_Y as well. Since the lines OP_X and OP_Y both pass through the same two points O and P, they are collinear. And since $OP_X = OP = OP_Y$, then $P_X = P_Y$.

A similar argument for ZZ' shows $OP_Z = OP = OP_X = OP_Y$. Therefore, $P_X = P_Y = P_Z = P'$ (Figure 4), which is what we wanted to prove, that is, all the perpendiculars from X', Y' and Z' meet at a single point P'.

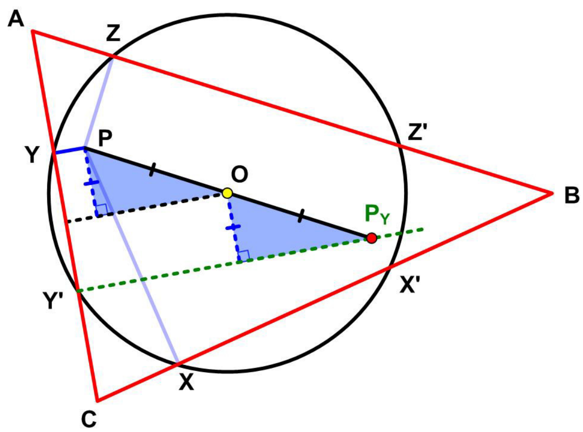

Figure 3

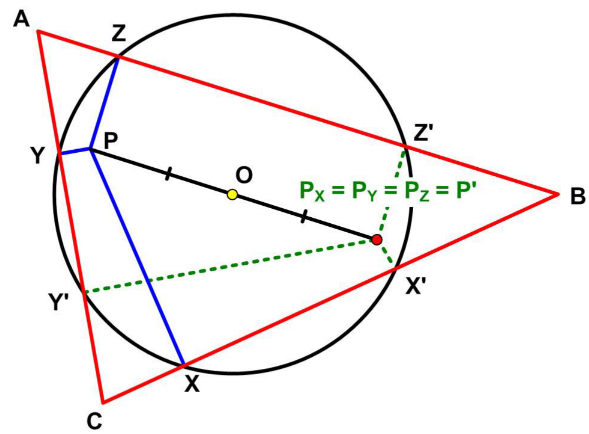

Figure 4

Crux Mathematicorum Solutions

Crux Mathematicorum had a slightly different approach for one solution and rather esoteric ideas for other solutions ([2]).

Solution by Ban Sokolowsky, Yellow Springs, Ohio.

We prove the following theorem, which shows that the property described in this problem holds for more general configurations, not just for triangles.

THEOREM. Let γ be a circle with centre O, let P *and* P' *be two points symmetric with respect to O, and suppose γ meets each of a set of lines L_i at points* X_i, X_i'. *Then the perpendiculars to each line L_i at* X_i *all pass through* P *if and only if the perpendiculars to L_i at* X_i' *all pass through* P'.

Proof. It is sufficient to show that the conclusion holds for an arbitrary line *L* of the set, which meets γ at X, X' (see Figure 5).

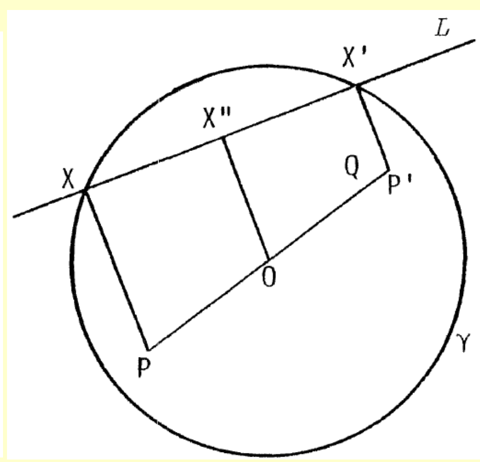

Figure 5

Suppose the perpendicular to *L* at X goes through P, and let the perpendicular to *L* at X' meet line PO in Q. If OX" ⊥ *L,* then XX" = X"X' and hence PO = OQ since PX || OX" ||QX'. Thus Q = P' and the perpendicular to *L* at X' goes through P'.

The converse can be proved by repeating the above argument with X, X' and P, P' interchanged.

Also solved by J.D. DIXON, *Haliburton Highlands Secondary School, Haliburton, Ont.;* CLAYTON W. DODGE, *University of Maine at Orono;* ROLAND H. EDDY, *Memorial University of Newfoundland;* SAHIB RAM MANDAN, *Indian Institute of Technology, Kharagpur, India; and the proposer.*

Editor's comment.

The proposer showed that when the perpendiculars at X, Y, Z meet at P (see Figure 6), then, as M ranges over the circle, the chords MM' ⊥ PM form an envelope of an ellipse inscribed in Δ ABC, which has P as one focus; and that the perpendiculars at M' all pass through P', which is the other focus of the ellipse. For details see (1).

Dodge showed that our problem is a nearly immediate consequence of a theorem about isogonal conjugate points, which can be found in (2).

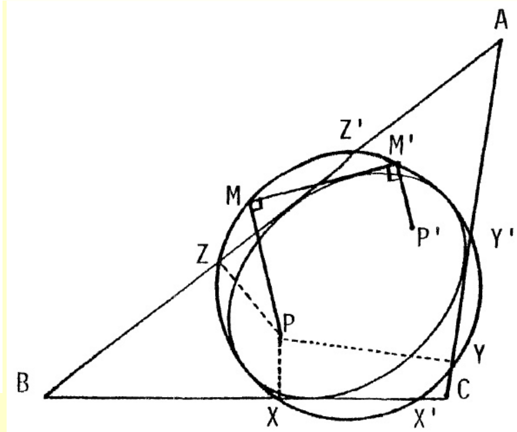

Figure 6

REFERENCES

(1) Dan Pedoe, *Geometry and the Liberal Arts*, Penguin Books, 1976, p. 207. (*If you can find a copy. This book was recently reviewed in* Crux Mathematicorum [1977: 7]*, and it was then thought that it would be distributed in Canada by Penguin Books Canada Ltd. I have since learned that copyright restrictions will make the expected Canadian distribution impossible, and that an American edition by St. Martin's Press is now in preparation. Canadians should have no difficulty getting a copy of the American edition when it is available.*)

(2) Nathan Altshiller Court, *College Geometry*, Barnes and Noble, 1952, p. 271, Theorem 641.

References

[1] Pedoe, Dan "Problem 206," *Crux Mathematicorum*, Vol.3 No.1 Jan, Canadian Mathematical Society, 1977. p.10

[2] Pedoe, Dan, "Problem 206 Solution," *Crux Mathematicorum*, Vol.3 No.5 Dec, Canadian Mathematical Society, 1977. p.143.