Diabolical Triangle Puzzle

28 January 2021

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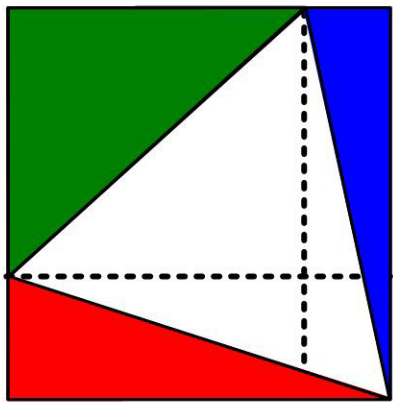


This simple-appearing problem is from the 17 August 2020 MathsMonday offering[1] by MEI, an independent curriculum development body for mathematics education in the UK.

The diagram shows an equilateral triangle in a rectangle. The two shapes share a corner and the other corners of the triangle lie on the edges of the rectangle. Prove that the area of the green triangle is equal to the sum of the areas of the blue and red triangles. What is the most elegant proof of this fact?

Since the MEI twitter page seemed to be aimed at the high school level and the parting challenge seemed to indicate that there was one of those simple, revealing solutions to the problem, I spent several *days* trying to find one. I went down a number of rabbit holes and kept arriving at circular reasoning results that assumed what I wanted to prove. Visio revealed a number of fascinating relationships, but they all assumed the result and did not provide a proof. I finally found an approach that I thought was at least semi-elegant.

My Solution

I tried a number of computational approaches, but they got too complicated and fraught with mistakes. Moreover, in the formulations I was using I was having a difficult time relating the blue and red triangles to the green triangle. Finally I resorted to vectors, which captured a way to relate the edges of the equilateral triangle with the edges of the rectangle via their horizontal and vertical components.

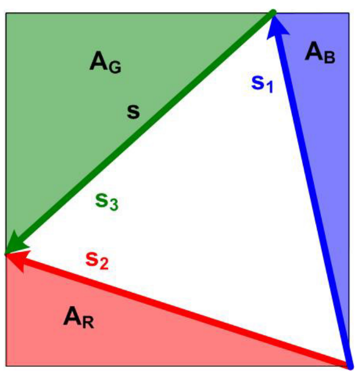


Figure 1

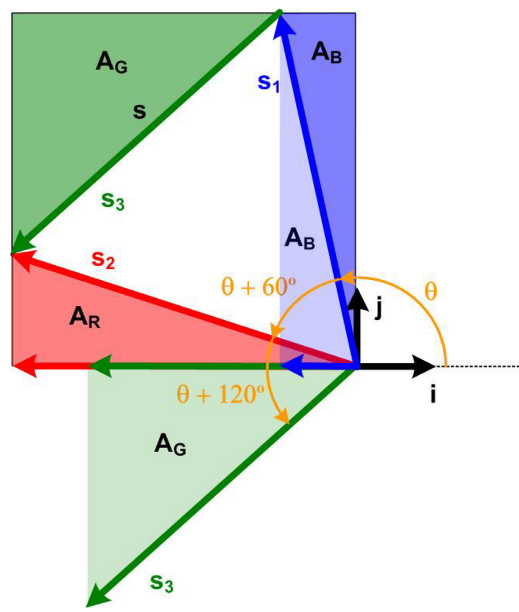


Figure 2

Let the areas of the three colored triangles be given by A_G, A_B, A_R as in Figure 1. Let $\mathbf{s}_k$ for $k = 1, 2, 3$, be the vectors of length s lying along each of the sides of the equilateral triangle. Figure 2 shows the key insight, that the areas of interest are between the side vectors and their horizontal components.

[1] https://twitter.com/MEIMaths/status/1295284235599859713

So for $k = 1, 2, 3$, the side vectors are given by

$$\mathbf{s}_k = s \cos(\theta + (k-1)60°)\,\mathbf{i} + s \sin(\theta + (k-1)60°)\,\mathbf{j}$$

Then horizontal components are the projections onto the horizontal axis, namely, for $k = 1, 2, 3$,

$$\mathbf{s}_k \cdot \mathbf{i} = s \cos(\theta + (k-1)60°)$$

Using the vector cross product, the areas are

$$A_B = \tfrac{1}{2}\,\mathbf{s}_1 \times (\mathbf{s}_1 \cdot \mathbf{i})\,\mathbf{i}, \quad A_R = \tfrac{1}{2}\,\mathbf{s}_2 \times (\mathbf{s}_2 \cdot \mathbf{i})\,\mathbf{i}, \quad A_G = \tfrac{1}{2}\,(\mathbf{s}_3 \cdot \mathbf{i})\,\mathbf{i} \times \mathbf{s}_3 = -\tfrac{1}{2}\,\mathbf{s}_3 \times (\mathbf{s}_3 \cdot \mathbf{i})\,\mathbf{i}$$

where we have reversed the order of the last product to maintain the same orientation of the angles between the vectors. Now for $k = 1, 2, 3$, the cross products are given by

$$\mathbf{s}_k \times \mathbf{i} = s \sin(\theta + (k-1)60°)\,\mathbf{j} \times \mathbf{i} = -\,s \sin(\theta + (k-1)60°)\,\mathbf{k}$$

Taking just the scalar component of the vector products and doubling the results, we have

$$2A_B = \mathbf{s}_1 \times (\mathbf{s}_1 \cdot \mathbf{i})\,\mathbf{i} = (\mathbf{s}_1 \cdot \mathbf{i})(\mathbf{s}_1 \times \mathbf{i}) = -\,s^2 \cos\theta \sin\theta = -\tfrac{1}{2}\,s^2 \sin 2\theta$$

$$2A_R = (\mathbf{s}_2 \cdot \mathbf{i})(\mathbf{s}_2 \times \mathbf{i}) = -\,s^2 \cos(\theta + 60°) \sin(\theta + 60°) = -\tfrac{1}{2}\,s^2 \sin 2(\theta + 60°)$$

$$2A_G = -\,(\mathbf{s}_3 \cdot \mathbf{i})(\mathbf{s}_3 \times \mathbf{i}) = s^2 \cos(\theta + 120°) \sin(\theta + 120°) = \tfrac{1}{2}\,s^2 \sin 2(\theta + 120°)$$

Therefore, using Figure 3

$$4A_B = -\,s^2 \sin 2\theta$$

$$4A_R = -\,s^2 \sin(2\theta + 120°)$$
$$= -\,s^2\,[\sin 2\theta\,(-1/2) + \cos 2\theta\,(\sqrt{3}/2)]$$

$$4A_G = s^2 \sin(2\theta + 240°)$$
$$= s^2\,[\sin 2\theta\,(-1/2) + \cos 2\theta\,(-\sqrt{3}/2)]$$

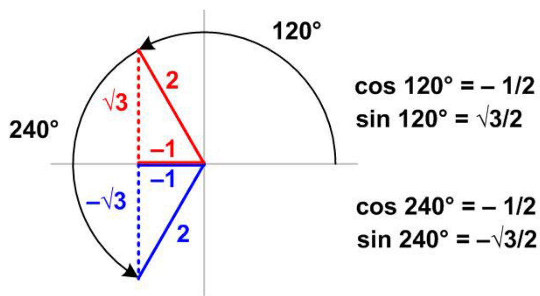


Figure 3

And from this we get our proof:

$$A_G = A_B + A_R$$

This approach does involve some trigonometry, but it is relatively straight-forward, though the proliferation of negative signs requires careful attention to get the answer to come out right. From a geometrical point of view the vector approach is clarifying, since it shows that as θ varies, moving vector $\mathbf{s}_1$, the other two vectors move in lock-step at 60º intervals, thus tying all the colored triangles together.

Rabbit Hole. Since I spent so much time on other approaches, I thought I would show the error of my ways in one of them.

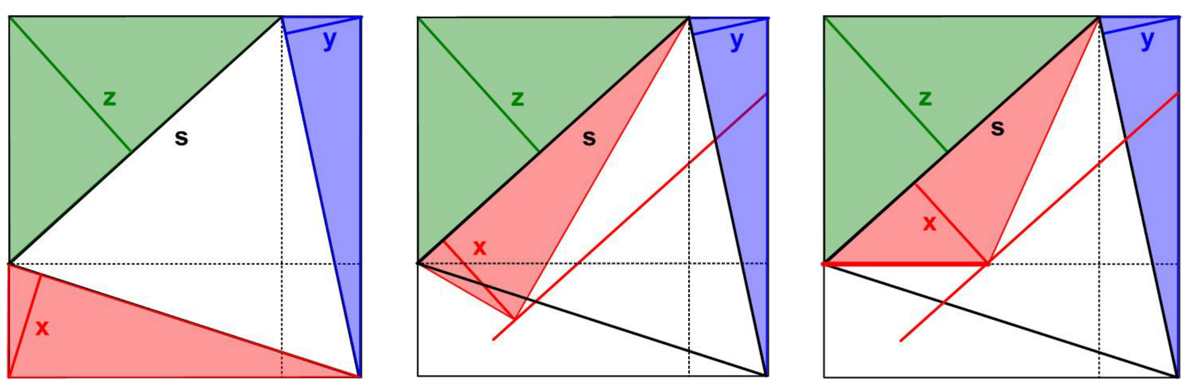


Figure 4 **Figure 5** **Figure 6**

The idea was to use geometric transformations to fill the duplicate image of the green triangle inside the equilateral triangle with the red and blue triangles. I indicated the altitudes *x, y, z* of the three triangles and the length *s* of the equilateral triangle (Figure 4). Then I rotated the red triangle 60º to lie against the green triangle along the side *s* (Figure 5). Using the area-preserving shear method, I moved the vertex of the red triangle along the parallel red line until it intersected the horizontal dashed line (Figure 6).

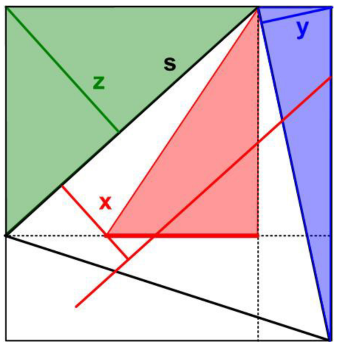


Figure 7

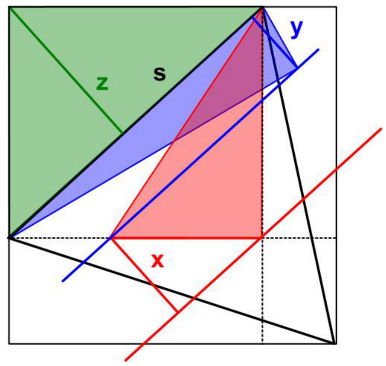


Figure 8

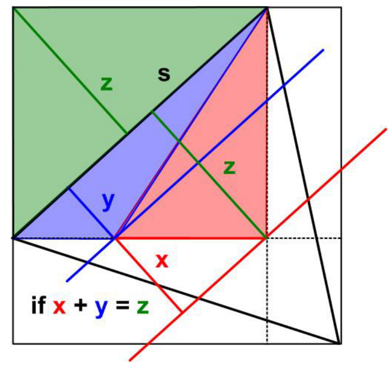


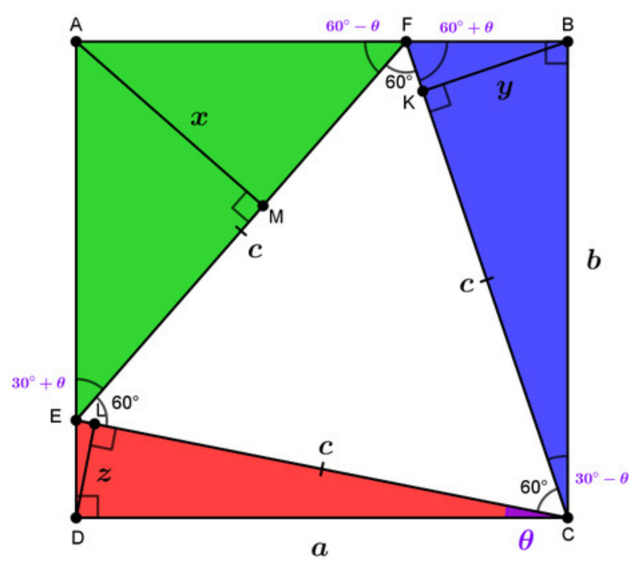
Figure 9

I then slid the base of the red triangle horizontally until it touched the vertical dashed line (Figure 7). I moved the red parallel line down until it intersected the vertex of the red triangle and rotated the blue triangle 60º clockwise to lie against the green triangle along the side *s* (Figure 8). Amazingly, the blue parallel line just intersected the end of the *x* altitude. I moved the vertex of the blue triangle along the blue line until it intersected the dashed horizontal line (Figure 9), and voila, the red and blue triangles filled the area of the green triangle.

However, as Figure 9 shows, the last two operations were tantamount to assuming $x + y = z$. Just because Visio showed it was so, that was not a proof. In fact, since $A_G = sz$, $A_B = sy$, and $A_R = sx$, the equation $x + y = z$ is equivalent to $A_G = A_B + A_R$, which was what I wanted to prove. To make the diagram work, I was tacitly assuming the answer I was trying to prove!

MEI Solutions

I did not find particularly elegant solutions to the problem at the MEI twitter page. I had tried some of these direct approaches as well, but my arithmetic failed me.



Nick Kalapodis[2] Solution

Let *a, b* be the dimensions of the rectangle and *c* be the side length of the equilateral triangle. We set $\theta = \angle ECD$ and let *x, y, z* be the heights of the triangles. Then

$z = \sin\theta \cdot a = \sin\theta\cos\theta \cdot c$

$y = \sin(30º - \theta) \cdot b = \sin(30º - \theta)\sin(30º + \theta) \cdot c$

and

$c = EM + MF = x \cdot \cot(30º + \theta) + x \cdot \cot(60º - \theta)$

[2] 7:35 PM · Aug 19, 2020 https://twitter.com/NickKalapodis/status/1296229357682397189

$\Rightarrow x = c \,/\, (\cot(30^\circ + \theta) + \cot(60^\circ - \theta))$

So it suffices to prove that

$$x = y + z$$

$$\frac{1}{\cot(30^\circ + \theta) + \cot(60^\circ - \theta)} = \sin(30^\circ - \theta)\sin(30^\circ + \theta) + \sin\theta\cos\theta$$

$$\frac{2}{\cot(30^\circ + \theta) + \tan(30^\circ + \theta)} = 2\sin(30^\circ - \theta)\sin(30^\circ + \theta) + 2\sin\theta\cos\theta$$

$$\frac{2\sin(30^\circ + \theta)\cos(30^\circ + \theta)}{\cos^2(30^\circ + \theta) + \sin^2(30^\circ + \theta)} = \sin(60^\circ - 2\theta) + \sin 2\theta$$

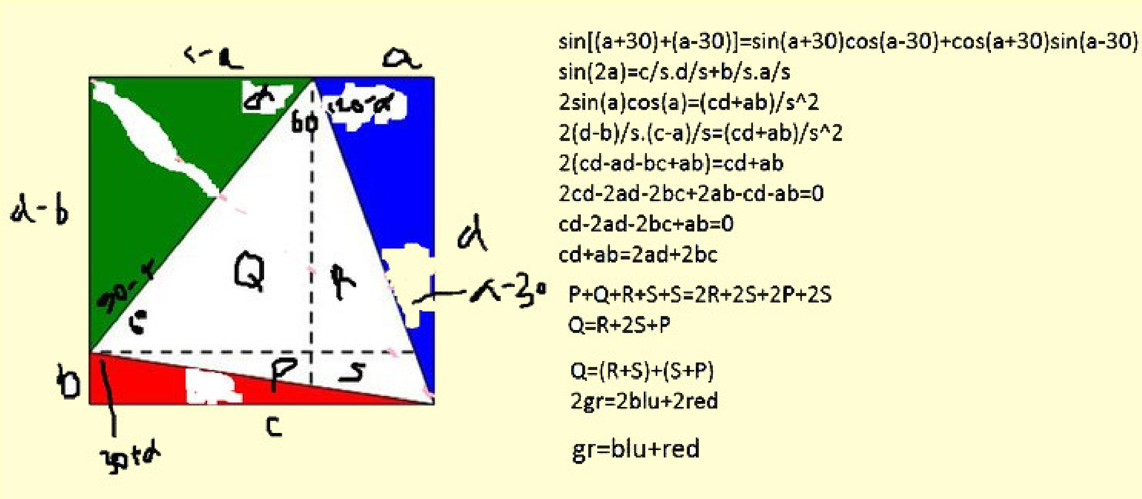
$$\sin(60^\circ + 2\theta) = \sin(60^\circ - 2\theta) + \sin 2\theta$$

$$\sin(60^\circ + 2\theta) - \sin(60^\circ - 2\theta) = \sin 2\theta$$

$$2\cos 60^\circ \sin 2\theta = \sin 2\theta$$

which is true.

Todi Nagawe[3] Solution



Robert Forrester[4] Solution

Adjust the diagram, so the red triangle disappears, and we have a symmetrical arrangement, where green = blue. The triangle and square then share the same lower side. Do the same and make the blue triangle disappear. Then green = red. :)

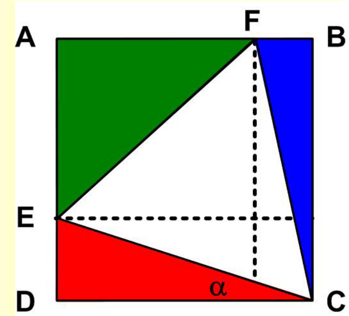
Clearly this last solution is only a partial one, since just two extreme cases were considered.

(Update 1/30/2021) MEI Item of the Month Solution[5]

I found the same problem under the MEI Item of the Month for August 2007.[6] The solution there is essentially the same as mine numerically, only it is not motivated by vectors.

[3] 10:34 AM · Aug 22, 2020 https://twitter.com/TodiNagawe/status/1297180223616630785
[4] 6:48 AM · Oct 9, 2020 https://twitter.com/RAnForr784/status/1314518110599249922
[5] https://mei.org.uk/files/miotm_solutions/miotm%20aug%2007.pdf
[6] https://mei.org.uk/month_item_07#aug

Let the side of the equilateral triangle be l and call the angle ECD α.
Let ΔCDE represent the area of triangle CDE.



You can then show that

$$\begin{aligned} DC &= l\cos\alpha \\ DE &= l\sin\alpha \end{aligned} \quad \Rightarrow \quad \Delta CDE = \frac{1}{2}l^2\cos\alpha\sin\alpha = \frac{1}{4}l^2\sin 2\alpha$$

$$\begin{aligned} BC &= l\cos(30-\alpha) \\ BF &= l\sin(30-\alpha) \end{aligned} \quad \Rightarrow \quad \Delta CBF = \frac{1}{2}l^2\cos(30-\alpha)\sin(30-\alpha) = \frac{1}{4}l^2\sin(60-2\alpha)$$

$$\begin{aligned} AE &= l\cos(30+\alpha) \\ AF &= l\sin(30+\alpha) \end{aligned} \quad \Rightarrow \quad \Delta EAF = \frac{1}{2}l^2\cos(30+\alpha)\sin(30+\alpha) = \frac{1}{4}l^2\sin(60+2\alpha)$$

Therefore

$$\begin{aligned} \Delta EAF - \Delta CBF &= \frac{1}{4}l^2\left[\sin(60+2\alpha)-\sin(60-2\alpha)\right] \\ &= \frac{1}{4}l^2\left[\sin 60\cos 2\alpha+\cos 60\sin 2\alpha-(\sin 60\cos 2\alpha-\cos 60\sin 2\alpha)\right] \\ &= \frac{1}{4}l^2\left[2\cos 60\sin 2\alpha\right] \\ &= \frac{1}{4}l^2\sin 2\alpha \\ &= \Delta CDE \end{aligned}$$

If instead you start by assigning lengths to the rectangles you might get into a mess.