Christmas Tree Puzzle

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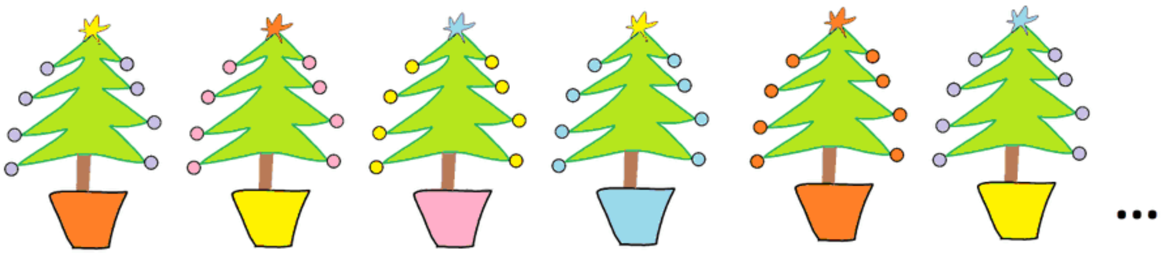

James Tanton has come up with another imaginative concrete problem[1] harboring a mathematical pattern.

60 trees in a row. Their stars are yellow, orange, blue, Y, O, B, Y, O, B, ... Their pots are orange, yellow, pink, blue, O, Y, P, B, O, Y, P, B, ... Their baubles are mauve, pink, yellow, blue, orange, M, P, Y, B, O, M, P, Y, B, O, ... Must there be an all yellow tree? All B? One with star = O, pot = O, baubles = M?

Solution

How many combinations of stars, pots, and baubles are there? 3 x 4 x 5 = 60. So with 60 trees we might think we have all combinations. But the decorated trees are in a specific order and so constrained. Does that mean we don't get all the combinations?

I am going to assign numbers to the colors as shown in the following table.

Stars	yellow	orange	blue		
	Y	O	B		
	1	2	0		
Pots	orange	yellow	pink	blue	
	O	Y	P	B	
	1	2	3	0	
Baubles	mauve	pink	yellow	blue	orange
	M	P	Y	B	O
	1	2	3	4	0

Let's number the trees 1, 2, …, 60 and lay down rows of star choices and pot choices, designating the colors by the numbers in the table. For simplicity, suppose there are only two star colors, numbered 1 and 0, and three pot colors, numbered 1, 2, 0:

Trees	**1**	**2**	**3**	**4**	**5**	**6**
Stars	1	0	1	0	1	0
Pots	1	2	0	1	2	0

[1] https://twitter.com/jamestanton/status/1342476455066759168 (25 December 2020)

The star color 1 (yellow) ends up being paired with each of the pot colors 1, 2, 0, as does the star color 0 (orange). So that covers all the possible 2 x 3 = 6 arrangements. Notice that the pot color paired with each star color is that star color's tree position modulo 3, that is, the remainder of dividing the tree position by 3.

But see what happens if we have 2 star colors and 4 pot colors. There should be 2 x 4 = 8 pairings to get all possible arrangements.

Trees	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
Stars	1	0	1	0	1	0	1	0
Pots	1	2	3	0	1	2	3	0

However, the star color 1 (yellow) is paired only with the pot colors 1 and 3, and not the colors 2 and 0. Similarly with the star color 0 (orange); it is only paired with the pot colors 2 and 4 and not 1 and 0. The patterns repeat their matching after only 4 trees. And this is because 2 divides 4, or more generally, 2 and 4 are not relatively prime. Another way to look at it is the tree positions for a given star color, modulo 4, only cycle through 2 pot color numbers instead of 4 numbers because 2 and 4 are not relatively prime.

If the stars had had 4 colors and the pots 6 colors, the alignment of the star and pot colors would repeat after 2 x 2 x 3 = 2 x 6 = 4 x 3 = 12 trees instead of the necessary 4 x 6 = 24 trees in order to get all possible arrangements. 4 and 6 are not relatively prime—they have a common factor 2.

But the original number of colors for the stars, pots, and baubles are 3, 4, and 5—all relatively prime with one another. So they will only line up with one another again after 3 x 4 x 5 = 60 trees, which covers all possible arrangements. (Notice the star color and pot color patters repeat after 3 x 4 = 12 trees, but their patterns do not repeat their match with the baubles colors until after 12 x 5 = 60 trees—12 and 5 are relatively prime.)

Trees	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**…**	**59**	**60**
Stars	1	2	0	1	2	0	1	2	0	1	2	0	1	…	2	0
Pots	1	2	3	0	1	2	3	0	1	2	3	0	1	…	3	0
Baubles	1	2	3	4	0	1	2	3	4	0	1	2	3	…	4	0

The star colors = Tree # mod 3, pot colors = Tree # mod 4, and bauble colors = Tree # mod 5. The only number simultaneously divisible by all three is 60.

$$60 \bmod 3 = 60 \bmod 4 = 60 \bmod 5 = 0$$

Therefore all possible arrangements of colors for stars, pots, and baubles will occur. And therefore the answer to the problem is all three color combinations will occur in the line of 60 trees. That is, there is an all yellow tree and an all blue tree and finally a tree with an orange star, orange pot, and mauve baubles. In fact,

	Tree #	**Star Color**	**Pot Color**	**Baubles Color**
All yellow	58	58 mod 3 = 1	58 mod 4 = 2	58 mod 5 = 3
All blue	24	24 mod 3 = 0	24 mod 4 = 0	24 mod 5 = 4
Orange star, orange pot, mauve baubles	41	41 mod 3 = 2	41 mod 4 = 1	41 mod 5 = 1