Curve Making Puzzle

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Here is a problem from *Five Hundred Mathematical Challenges* ([1]) that I indeed found quite challenging.

Problem 235. Two fixed points A and B and a moving point M are taken on the circumference of a circle. On the extension of the line segment AM a point N is taken, outside the circle, so that $MN = MB$. Find the locus of N.

Since one of the first hurdles I faced with this problem was trying to figure out what type of shape was being generated, I thought I would omit my usual drawings illustrating the problem statement. There turned out to be a lot of cases to consider, but the result was most satisfying. I also included the case when N is inside the circle. Again Visio was my main tool to handle all the examples with the concomitant requirement to prove whatever Visio suggested.

My Solution

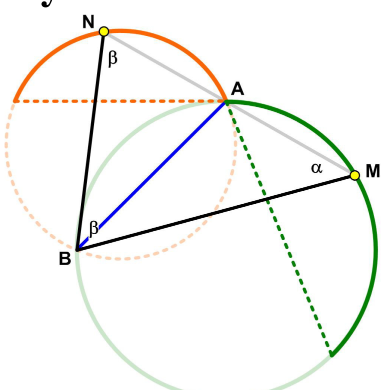

Figure 1 Case N A M

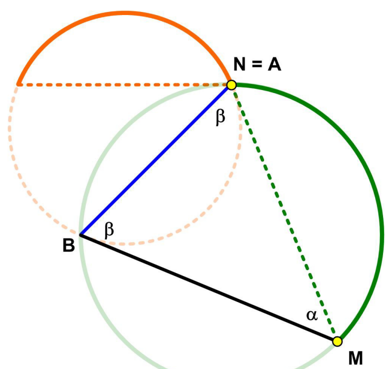

Figure 2 Case N=A M

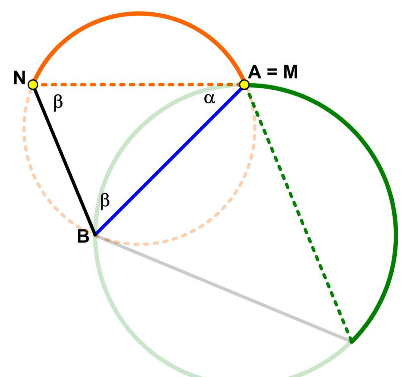

Figure 3 Case N A=M

First Circle. We begin with the line through A and M having M on the right and N on the left (Figure 1). As M moves along the (green) circle, the angle α is an inscribed angle of the circle subtending the same arc and is therefore constant. Given that the distances MN = MB, the triangle BMN is isosceles and so the base angles are equal and given by β. Therefore all the triangles BMN are similar. As the point M on BM traces out the arc of the green circle, the point N on BN traces out a similar circular arc shrunk by the ratio BM : BN and rotated counter-clockwise by the angle β (Figure 1, Figure 2, Figure 3).

Claim. *The center O of the orange circle lies on the original green circle* (Figure 4). Notice that the point A on both the green and orange circles lies on the perpendicular bisector of BN through the center O of the orange circle, since MB = MN. And then O also lies on the perpendicular bisector of AB. But the perpendicular bisector of AB also passes through the center of the green circle O'. Moreover, line BN is parallel to line AM', since

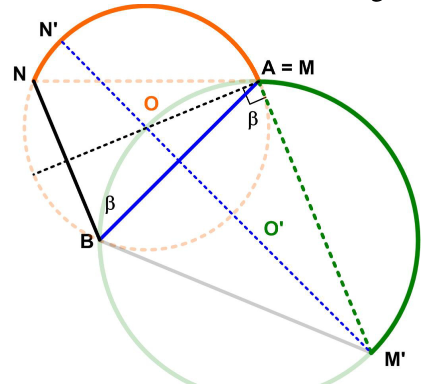

Figure 4 Circle Properties

the alternating interior angles β are equal. Therefore the diameter AO of the orange circle being perpendicular to BN means it is also perpendicular to the line AM', that is, the angle at A, OAM' is a right angle. That implies that the center of the orange circle O must also lie on the end of the diameter of the green circle.

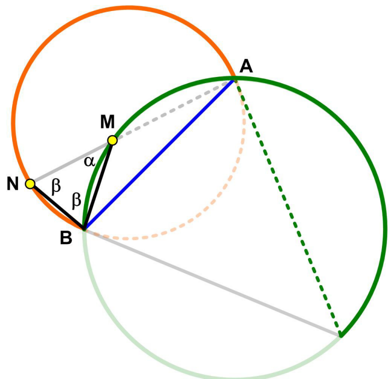

Figure 5 Case N M A

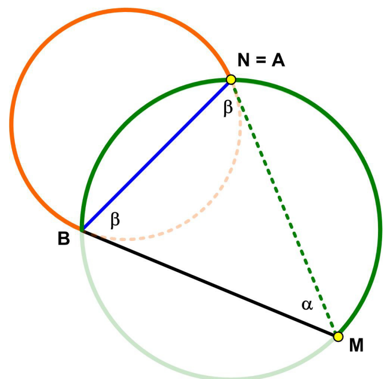

Figure 6 Case A=N M

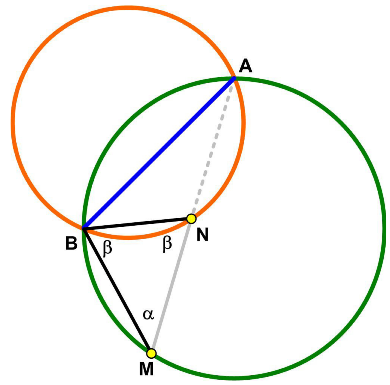

Figure 7 Case A N M

We can extend the path of the orange circle further. Figure 5 shows what happens to the situation in Figure 3 when we move the point M to the left of the point A and between A and N. The similar triangle BMN continue to shrink and the point N continues to trace out shrunken and rotated image of the green circle along which M is moving. This continues until the triangles shrink to a point at B where M, N, and B all coincide.

Return to the end case in Figure 2 (Figure 6). *If we allow N inside the green circle*, then it will continue to generate a path along the orange circle as M moves along the original green circle (Figure 7) until the similar triangles shrink to a point where M, N, and B all coincide.

Second Circle. There is yet another set of circumstances that fulfill the statement of the problem. Rather than have the point N to the left along the line through A and M, we shall consider it to the right. Figure 8 shows the initial situation when M coincides with the point A. I have included the previous case for the first (orange) circle with the original N represented by N'. Since N', A=M, and N all lie on the same straight line, angle α = γ, which implies γ is also constant as M traces the green circle. Furthermore, MN = MB implies BMN is again an isosceles triangle with equal base angles δ. Finally, β + δ + (β + δ) = 180° implies β + δ = 90°. Figure 9 shows that as the point M on

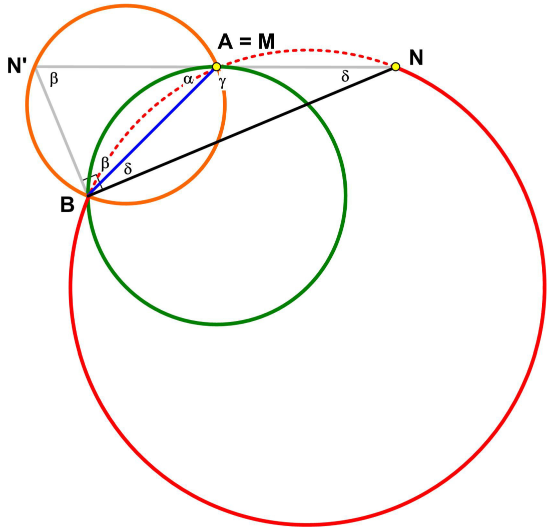

Figure 8 Case A=M N

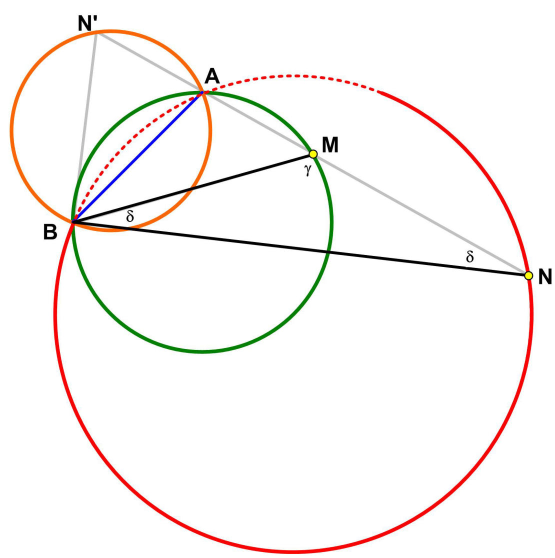

Figure 9 Case A M N

BM traces out the arc of the green circle, the point N on BN traces out a similar circular arc expanded by the ratio BN : BN and rotated clockwise by the angle δ.

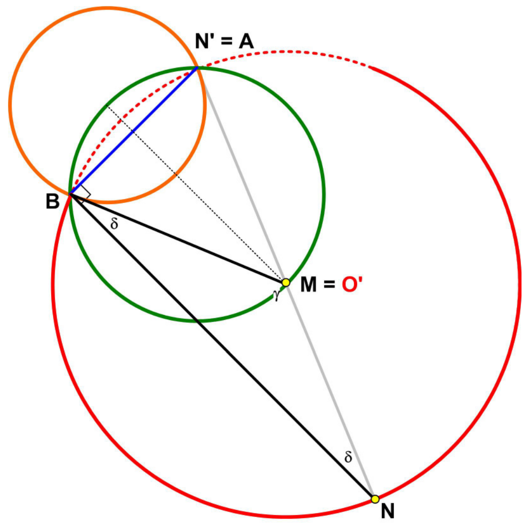

Figure 10 Case A M N

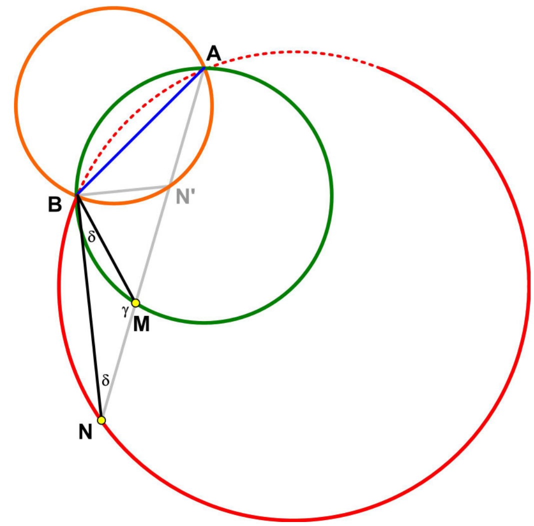

Figure 11 Case A M N

Figure 10 shows what happens when M reaches the vertex of the isosceles triangle BMA. Since ABN is a right angle, AN is a diameter of the red circle passing through M. But since M is also on the perpendicular bisector of AB, *M = O' is the center of the red circle.*

Figure 11 shows the situation as M continues along the original green circle until it (and N) coincide with B. Figure 11 also shows that there are some (dotted) sections of the red circle still not traversed by N.

If we return to the initial situation shown in Figure 8 and now move M to the left of A, then N traverses the remainder of the red circle to the right of A until it reaches A (Figure 12). MN = MB implies M is on the perpendicular bisector of AB and therefore the diameter of the green circle. Therefore the angle MAO' is a right angle and AM is tangent to the red circle. Moreover, MB = MN' means M is also on the perpendicular bisector of BN' and so the center O of the orange circle. Thus *the two generated circles intersect orthogonally*.

Finally, Figure 13 shows that as M continues along the green circle to B, *if N is allowed inside the*

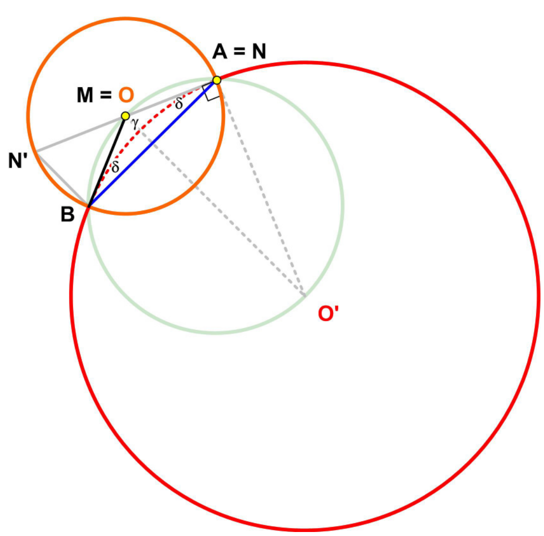

Figure 12 Case M N=A

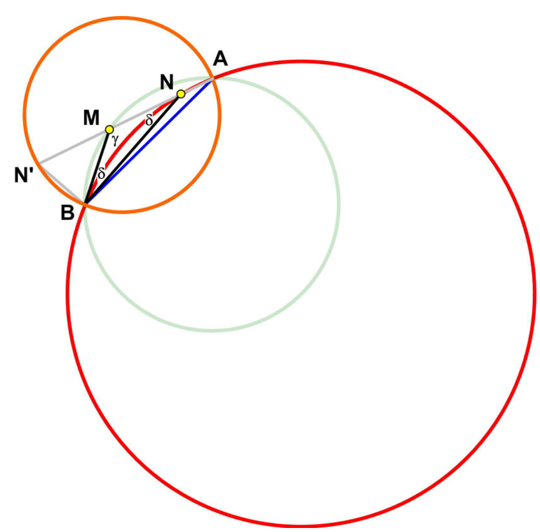

Figure 13 Case M N A

green circle, then N continues to trace the red circle also to B. Thus the final solution is shown in Figure 14, if N is allowed inside the green circle, and Figure 15, if N is restricted to outside the green circle, as originally stated in the problem. (Also notice that N is not on the "extension" of the line segment AM if and only if N is inside the green circle. See Figure 7 and Figure 13.)

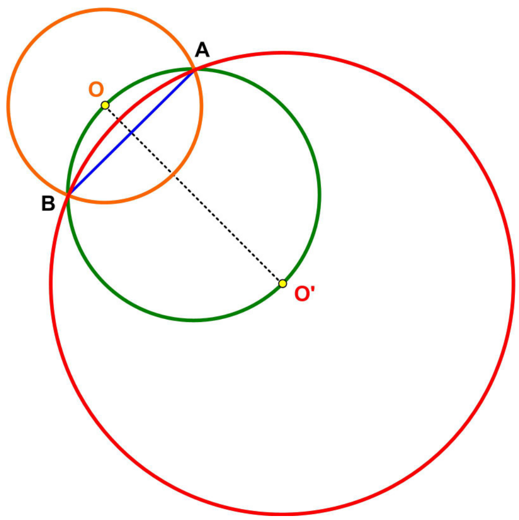

Figure 14 Final Solution (N unrestricted)

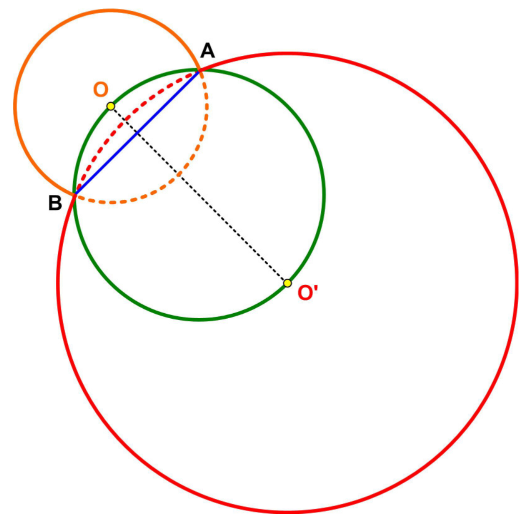

Figure 15 Final Solution (N outside Circle)

Challenges Solution

Naturally, sight unseen, I chose the opposite orientation for the line AB. If you flip Figure 16 around the diagonal from upper left to lower right, then you get my figures (or flip my figures to get the *Challenges* figures).

Problem 235. Referring to Figure 16, we see that, since ΔBMN is isosceles, $\angle BNM = ½\angle BMA$, which is a constant angle. Hence N must lie on a circle passing through points *A, B*. The center of this circle will be at that point *O* for which $\overline{AO} = \overline{BO} = \overline{OQ}$. However, the locus will include only part of the circle, namely the arc from *B* to the point *P* such that $\overline{AP} = \overline{AB}$ and *AP* is tangent to the circle *AOB*. For if *N* lies on the arc, then $\angle BNA = ½\angle AOB = ½\angle AMB$ whence $\angle BNM = \angle MBN$ and $\overline{BM} = \overline{BN}$.

Taking into account the possibility that *M* is on either side of *AB,* we see that the locus of *N* consists of two circular arcs, both emanating from *B,* both part of circles through *A* and *B* with centers at the ends of the diameter bisecting *AB* and both lying on the same side of the tangent to the given circle through *A*.

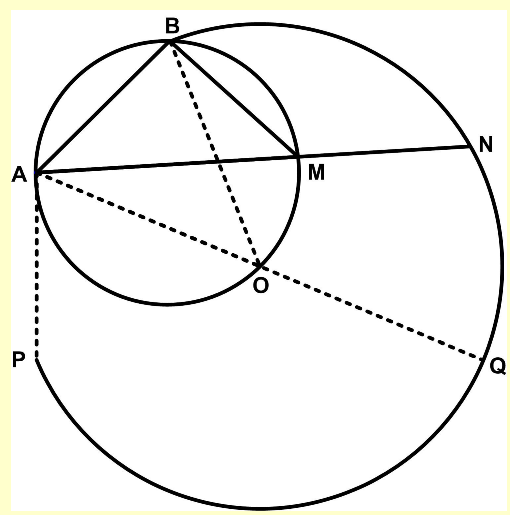

Figure 16 *Challenges* Solution

Question. How would you interpret the remainder of the circles?

Comment. First, I admit I don't quite follow the argument in the first paragraph that eliminates N on the arc from P to A to complete the circle. In the second paragraph the *Challenges* solution does not show my first orange circle, though it describes it in words. The thing I don't understand is the restriction "and both lying on the same side of the tangent to the given circle through *A*."

Clearly, it is a lot of work to consider all the situations and cases, so I understand why *Challenges* did not provide diagrams and explanations for everything, but the absence makes it difficult for me to figure out where our solutions would differ. Also, they neglected to mention the two circles intersect orthogonally, which I thought was pretty cool. The entire result was rather surprising at first, based on the very indirect method of generating the curves.

References

[1] Barbeau, Edward J., Murray S. Klamkin, William O. J. Moser, *Five Hundred Mathematical Challenges*, Spectrum Series, Mathematical Association of America, Washington D.C, 1995