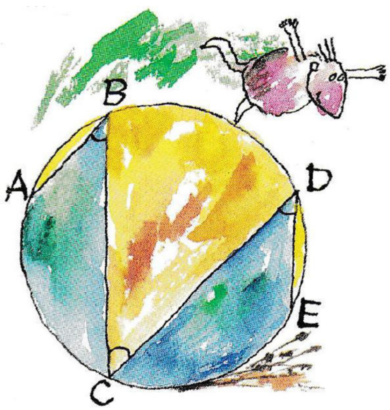
Circle-Halving Zigzag Problem

26 July 2020

Jim Stevenson



Pavel Chernusky

This is another delightful Brainteaser from the *Quantum* math magazine ([1]).

All the vertices of a polygonal line ABCDE lie on a circumference (see the figure), and the angles at the vertices B, C, and D are each 45°.

Prove that the area of the blue part of the circle is equal to the area of the yellow part. (V. Proizvolov)

I especially liked this problem since I was able to find a solution different from the one given by *Quantum*. Who knows how many other variations there might be.

My Solution

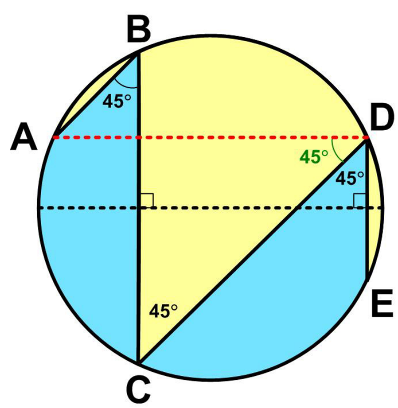


Figure 1 Step 1

First, draw the perpendicular bisector of the line BC (black dotted line in Figure 1). This line will pass through the center of the circle and is therefore a diameter of the circle. Since DE is parallel to BC, its perpendicular bisector through the center coincides with the same diameter.

Draw a line from A to D (red dotted line in Figure 1). Since the inscribed angle in the circle at D subtends the same arc as the inscribed angle at B, it is also 45°. Therefore line AD makes a right angle with DE and so is parallel to the diameter of the circle.

Now this is the crucial step: flip the zigzag line around the diameter as shown in Figure 2. Then we begin pairing yellow and blue sections of the circle that have the same areas. First the yellow curved section along BD is congruent with the blue curved section along CE, and so their areas are equal. Next, the vertical yellow sliver of the circle at DE is congruent with the vertical blue sliver at A, and so the areas are equal. Finally, take the small blue triangle at DE and translate it to A on the other side of the circle. This fills in the corner of the square with diagonal BC (after removing the vertical blue sliver). The

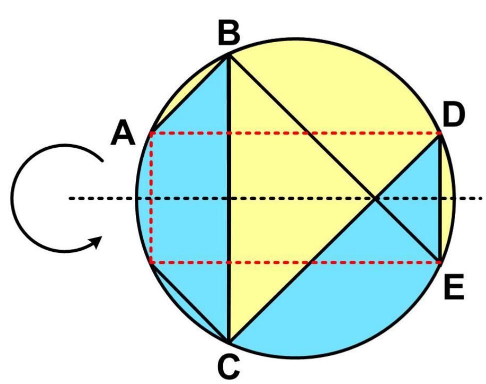


Figure 2 Step 2

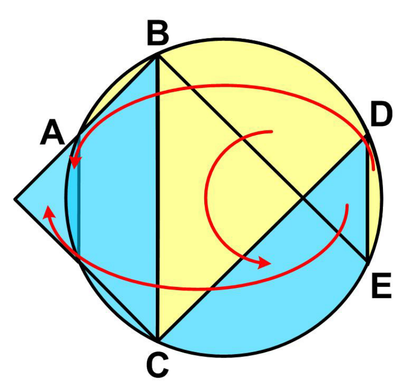


Figure 3 Step 3: Solution

square is made up of two congruent triangular halves of the square, and so their areas are equal. Therefore we have shown that every yellow section of the circle is paired with a blue section with the same area. And so the zigzag line cuts the circle into two equal areas.

Quantum Solution

Here is the *Quantum* solution:

Join the center O of the circle to all the five given points (Figure 4). Then the angles AOC, BOD, EOC are right angles, because each of them is a central angle that intercepts the same arc as the corresponding inscribed angle measuring 45°. It follows that the center O lies inside the angle BCD, and the yellow area is divided by the radii drawn above into five pieces: two circular segments AB and DE, circular sector BOD, and two triangles OBC and ODC. Note that

$$\angle DOC = 180° - \angle AOB,$$

$$\angle BOC = 180° - \angle DOE.$$

Therefore, the area of triangle ODC, which is equal to

$$\tfrac{1}{2}\, OC^2 \sin\angle DOC,$$

is equal to the area of triangle AOB ($\sin\angle DOC = \sin\angle AOB$), and area($\Delta$OBC) = area($\Delta$ODE).

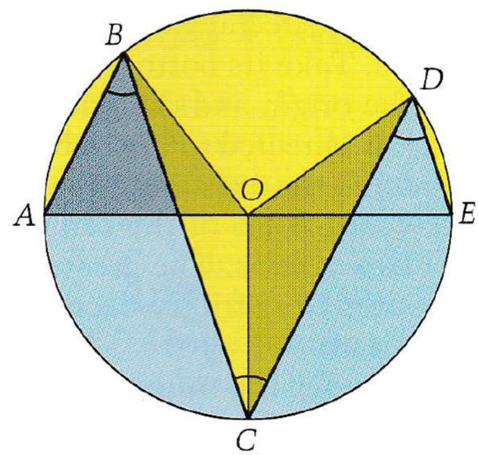


Figure 4 ***Quantum*** **Solution**

Now, if we replace the triangles ODC and OBC in the yellow figure by triangles OAB and ODE with the same areas, respectively, we'll turn this figure into the semicircle ABCDE without changing its area, which, consequently, equals the blue area. (V. Dubrovsky)

References

[1] "Circle-Halving Zigzag", in "Brainteasers" *Quantum Magazine*, Vol.4 No.4, National Science Teachers Assoc., Springer-Verlag, Mar-Apr 1994 p.15