Pool Paving Problem

24 September 2020

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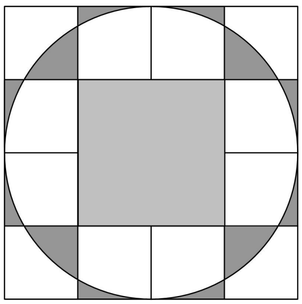
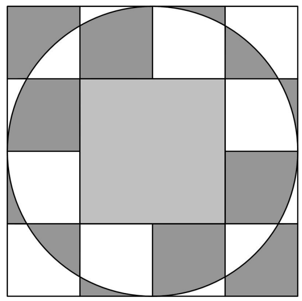


(a) Paving Design 1



(b) Paving Design 2

In my search for problems I decided to purchase Dan Griller's GCSE problem book ([1]) mentioned in the Cube Roots Problem.[1] I am still a bit confused about the purpose of the GCSE exam and who it is for, since the other problems in Griller's book are often as challenging or more so than the cube roots problem. It is hard to believe students not pursuing college level degrees could solve these problems. (Grades 8 and 9 referred to in the subtitle of the book must indicate something other than US grades 8 and 9, since the exams are aimed at 16 year-olds, not 13 and 14 year-olds.)

Supposedly the problems in Griller's book are nominally arranged in increasing order of difficulty from problem 1 to problem 75. However it seemed to me that there were challenging problems scattered throughout and the last problem was not all that much harder than earlier ones. And many of them had a whiff of Coffin Problems—they seemed impossible at first (Problem 44: Construct a 67.5° angle! Again, it's not that hard once you see the idea.). I don't know how many problems are on the exam or how long the exam is, but anyone taking a timed exam does not have the leisure to mull over a problem. The student only has a few minutes to come up with an approach and clever insights are rare under the circumstances. Anyway, here is the last problem in the book.

Problem 75. A square pond of side length 2 metres is to be surrounded by twelve square paving stones of side length 1 metre.

(a) The first design is constructed with a circle whose centre coincides with the centre of the pond. Calculate exactly the total dark grey area for this design.

(b) The second design is similar. Calculate exactly the total dark grey area for this second design.

My Solution for Paving Design 1

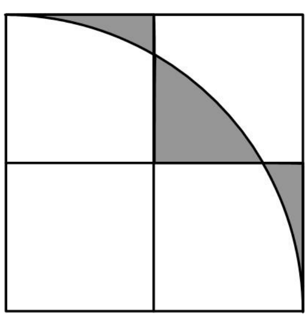


Figure 1 Area A_0

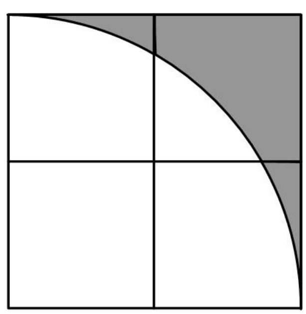


Figure 2 Area A_1

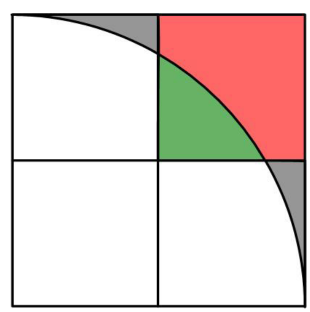


Figure 3 Area A_2 of Green Region

[1] http://josmfs.net/2019/06/12/cube-roots-problem/

As shown in Figure 1, we will compute the shaded area A_0 in a quarter of the paving stones and multiply by 4 to get the total. To accomplish this we first find the shaded area A_1 shown in Figure 2.

$$A_1 = \text{area square} - \text{area of quarter circle} = 4 - \pi.$$

Now we subtract the (red) shaded area in the upper right square and add the (green) shaded curved region as follows.

First, we need to compute the area A_2 of this green region. From Figure 4 we see that the shaded sector has an angle of $30° = \pi/6$ and so is a third of the area of quarter circle or

$$\text{Area sector} = \pi/3$$

From Figure 5 by the symmetries of the problem we see that the colored triangles are congruent and have base $\sqrt{3} - 1$ and altitude 1. Therefore the sum of their areas is $\sqrt{3} - 1$. We now subtract this from the sector to get the area A_2 of the green region

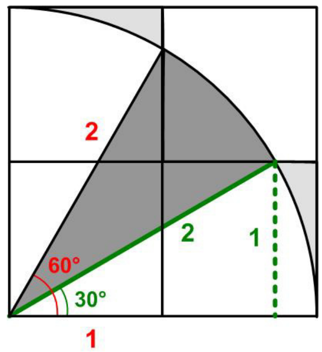


Figure 4

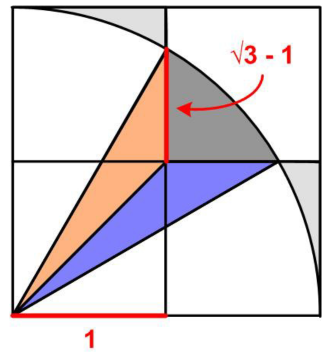


Figure 5

$$A_2 = \text{area of sector} - \text{area of triangles} = \pi/3 - \sqrt{3} + 1$$

Therefore the desired area A_0 is, subtracting the area of 1 m^2 of the upper right square and adding twice the area A_2 of the green region,

$$A_0 = A_1 - 1 + 2A_2 = 4 - \pi - 1 + 2(\pi/3 - \sqrt{3} + 1) = 5 - 2\sqrt{3} - \pi/3$$

So the final area for Paving Design 1 is

$$4\,A_0 = 20 - 8\sqrt{3} - 4\pi/3 \text{ meters}^2$$

My Solution for Paving Design 2

Compared to the first problem this solution is trivial.

We shade the areas A_0 from problem (a) in rose and rotate them 90° counter-clockwise, and voila! They exactly fill in the blank areas to yield 6 solid paving stones with an area of 6 meters^2.

I tried this idea in problem (a) for some time, and then gave up and solved it with the brute-force method.

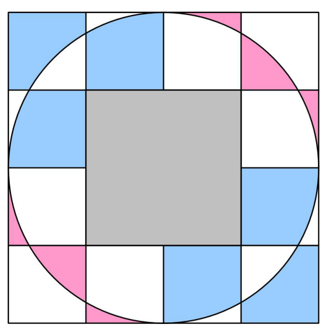


Figure 6

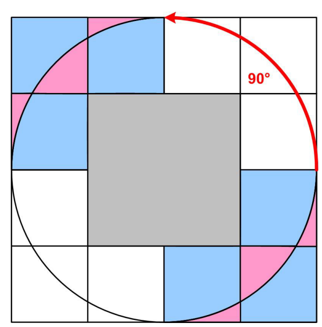


Figure 7

Griller Solution for Paving Design 1

Add the lines to the diagram as shown in Figure 8. Consider triangle ABE.

AB = 1

AE = 2 (radius)

ABE = 90°

So $$BE = \sqrt{2^2 - 1^2} = \sqrt{3} \quad \text{(Pythagoras)}$$

And hence the area of ABE is

$$\frac{1}{2}\times 1\times\sqrt{3}=\frac{\sqrt{3}}{2}$$

We also have

$$\cos BAE = ½$$

and so

$$BAE = 60° \text{ and } EAG = 30°$$

This means that the sector EAG represents one twelfth on the entire circle, so its area is

$$\frac{1}{12}\times\pi\times 2^2=\frac{\pi}{3}$$

The grey region EFG combines with the sector EAG and triangle ABE to cover exactly two paving stones. So

$$\text{Area (EFG)} = 2-\frac{\pi}{3}-\frac{\sqrt{3}}{2}$$

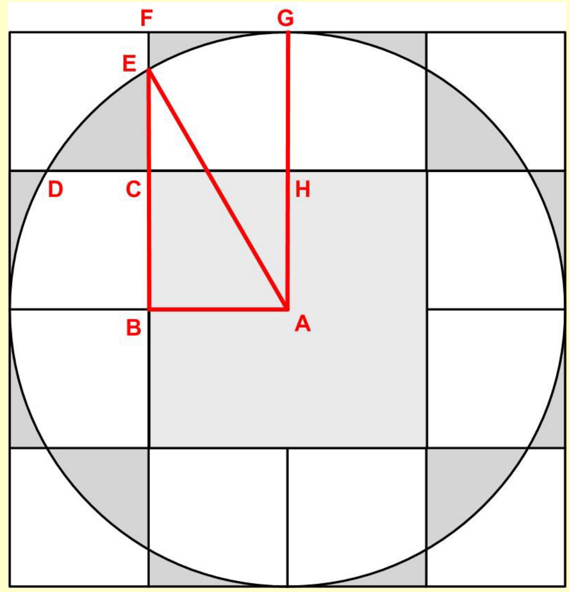


Figure 8 Griller Solution

and then

$$\text{Area (CEGH)} = 1 - \text{area (EFG)} = \frac{\pi}{3}+\frac{\sqrt{3}}{2}-1$$

Next, the grey region CDE combines with two white regions like CEGH and a single square of side 1 to cover a quarter of the circle. So

$$\text{Area (CDE)} = \frac{1}{4}\times\pi\times 2^2 - 1 - 2\times\text{Area (CEGH)} = 1+\frac{\pi}{3}-\sqrt{3}$$

Finally, the total grey area comprises 8 regions like EFG and 4 regions like CDE. Thus the final answer is

$$8\left(2-\frac{\pi}{3}-\frac{\sqrt{3}}{2}\right)+4\left(1+\frac{\pi}{3}-\sqrt{3}\right)=\boxed{20-\frac{4\pi}{3}-8\sqrt{3}\ m^2}$$

Griller Solution for Paving Design 2

Imagine drawing a vertical line down the centre of the diagram [(b) Paving Design 2]. The design is identical on either side of this line, except that any grey regions on the left are white on the right, and vice versa. Hence, exactly half of the paving is grey, and so the total grey paved area is 6 m^2.

References

[1] Griller, Daniel, *Problem Solving in GCSE Mathematics, Questions for pupils targeting Grades 8 and 9*, Rational Falcon, 2018.