Pointing Fingers

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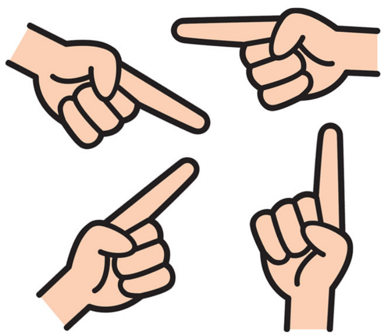

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Here is a nice logic puzzle from 2014 *Futility Closet* ([1]).

Only one of these statements is true. Which is it?

A. All of the below
B. None of the below
C. One of the above
D. All of the above
E. None of the above
F. None of the above

My Solution

The easiest way to approach this problem is to translate it into symbolic logic where the statements are represented by the capital letters. We shall review some preliminary aspects of symbolic logic (the propositional calculus) first.

Symbolic Logic Preliminaries

Negation operator "~"	Given a statement A, **~A** is its negation. That is, if A is true, ~A is false, but if A is false, ~A is true.
Conjunction operator "∧"	Given two statements A and B, the composite statement **A∧B** is true if and only if both A and B are true. It is false if either or both are false.
Disjunction operator "∨"	Given two statements A and B, the composite statement **A∨B** is true if and only if either A or B (or both) is true. It is false if both are false.
Implication operator "⇒"	Given two statements A and B, the composite statement **A ⇒ B** is true if and only if when A is true, then B is true. That is, it can never be that A is true and B is false.
Equivalence operator "≡" or "⇔"	Given two statements A and B, the composite statement **A ≡ B** is true if and only if both A and B are true or both A and B are false. That is, A is true if and only if B is true. This is equivalent to (A ⇒ B) ∧ (B ⇒ A).

An invaluable tool in the propositional or statement calculus is the truth table. The easiest explanation is to provide some examples. We assign to the statements all possible combinations of true (T) and false (F) and see what the corresponding truth values are for the composite statements.

~A

A	~A
T	**F**
F	**T**

A ∧ B

A	B	A ∧ B
T	T	**T**
F	T	**F**
T	F	**F**
F	F	**F**

A ∨ B

A	B	A ∨ B
T	T	**T**
F	T	**T**
T	F	**T**
F	F	**F**

A ⇒ B

A	B	A ⇒ B
T	T	**T**
F	T	**T**
T	F	**F**
F	F	**T**

A ≡ B

A	B	A ≡ B
T	T	**T**
F	T	**F**
T	F	**F**
F	F	**T**

We can use truth tables to establish the logical equivalences given in the definitions of the composite statements. For example, (A ∧ B) ≡ ~(~A ∨ ~B), which in words means both A and B are true if and only if it is not the case the either A is false or B is false (or both are false). We will condense the table a bit for brevity.

(A	**∧**	**B)**	**≡**	**~(**	**~(**	**A)**	**∨**	**~(**	**B))**
(1)	(2)	(1)	**(6)**	(5)	(3)	(1)	(4)	(3)	(1)
T	T	T	**T**	T	F	T	F	F	T
F	F	T	**T**	F	T	F	T	F	T
T	F	F	**T**	F	F	T	T	T	F
F	F	F	**T**	F	T	F	T	T	F

Steps	**Actions**
(1)	On the left and the right, fill in all combinations of T and F for statements A and B.
(2)	On the left, evaluate the conjunction A ∧ B for the corresponding values of A and B.
(3)	On the right, evaluate the negations ~A and ~B.
(4)	On the right, evaluate the disjunction ~A ∨ ~B
(5)	On the right, evaluate the negation ~(~A ∨ ~B)
(6)	Finally, from (2) and (5) evaluate the equivalence to show a tautology, that is, the left and right sides have the same truth values for corresponding values of A and B, and so are logically equivalent.

Note that since the negation of a negation restores the original statement, we have ~(~A) ≡ A. So the above equivalence can be written

$$\sim(A \wedge B) \equiv (\sim A \vee \sim B)$$

This is one of the De Morgan "Laws". The other is

$$\sim(A \vee B) \equiv (\sim A \wedge \sim B)$$

Problem Solution

Now to return to the original problem.

Symbolic Logic Stmt Translation	**Negation of Statement via De Morgan Laws**	**Implications**	
		If B is true, …	∴ B false, ~B true
A ≡ B∧C∧D∧E∧F	~A ≡ ~B∨~C∨~D∨~E∨~F		~B true implies ~A true.
B ≡ ~C∧~D∧~E∧~F	~B ≡ C∨D∨E∨F	B true implies ~C true.	
C ≡ A∨B	~C ≡ ~A∧~B	B true implies C true. Contradiction. **∴ B false.**	~A and ~B true implies ~C true.
D ≡ A∧B∧C	~D ≡ ~A∨~B∨~C		~B true implies ~D true.
E ≡ ~A∧~B∧~C∧~D	~E ≡ A∨B∨C∨D		~A, ~B, ~C, and ~D true implies E true.
F ≡ ~A∧~B∧~C∧~D∧~E	~F ≡ A∨B∨C∨D∨E		E true implies ~F true.

Therefore statement E is the only true statement, all the others are false.

Futility Closet Solution

Consider them in turn. A can't be true because some of the other statements contradict one another (for example, C and D[1]). B can't be true because it implies that C is also true.[2] Since A and B are false, C is also false, and likewise D. But E is true, and this makes F false. So the answer is E.[3]

From *Crux Mathematicorum*, 1979 [Vol.5 No.3 Mar], p.83, Problem 357 [Solution], via Ross Honsberger's *More Mathematical Morsels*, 1991.[4]

Crux Mathematicorum Solution

Here is the original solution to the originally posed problem.

The truth of	is then contradicted by	therefore
A or F	E[5]	A and F are false
C	B[6]	C is false
B	D[7]	B is false
D	The falseness of A, B, C[8]	D is false

So the only answer that can be true is E, provided the truth of E leads to a consistent set of truth values and the falsity of E does not.[9] This is in fact the case,[10] so E is the only true answer.

Comment

As should be clear from my notes on the other two solutions, I believe they have skipped steps or made unjustified statements.

References

[1] "Pointing Fingers", *Futility Closet*, 17 June 2014.
(https://www.futilitycloset.com/2014/06/17/pointing-fingers/)

1 JOS: How do C and D contradict each other if A is true? Nor do they contradict A if they are true.

2 JOS: Why can't C be true?

3 JOS: Certainly, once you have A, B, C, and D false the rest follows.

4 JOS: The original problem was proposed in *Crux Mathematicorum*, 1978, Vol.4 No.6 Jun, p.160, Problem 357.

5 JOS: How do you know E is true at this stage?

6 JOS: How do you know B is true at this stage? In fact, you claim it is false in the next line.

7 JOS: How does the statement D ≡ A∧B∧C contradict that B is true? It actually implies B is in fact true.

8 JOS: This at least is a correct deduction.

9 JOS: Whoa! That is the whole point of the proof!

10 JOS: Asserting the truth does not make it true. A proof is needed.