Another Cube Slice Problem

4 August 2020

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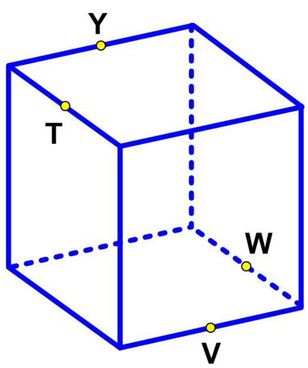

This is a problem from the UKMT Senior Challenge for 2019. (It has been slightly edited to reflect the colors I added to the diagram.)

23. The edge-length of the solid cube shown is 2. A single plane cut goes through the points Y, T, V and W which are midpoints of the edges of the cube, as shown.

    What is the area of the cross-section?

    A $\sqrt{3}$ B $3\sqrt{3}$ C 6 D $6\sqrt{3}$ E 8

My Solution

As a first approximation, the plane slicing the cube will pass through the points Y, T, V and W as shown in Figure 1. Taking a side view (Figure 2) shows that the plane passes through the midpoint of the cube edge at U. Therefore, it is clear that the intersection of the plane with the cube yields a hexagon (Figure 3). Since all the points are midpoints of their respective edges, they lie one unit from the corners of the cube and so the edges of the hexagon connecting them must be $\sqrt{2}$ units long (Figure 4). Furthermore, by the rotational symmetry of the cube and hexagon around the (red) diagonal perpendicular to the hexagon, we see that all the interior angles of the hexagon are equal and so it is regular.

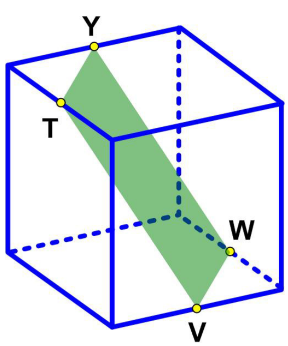

Figure 1

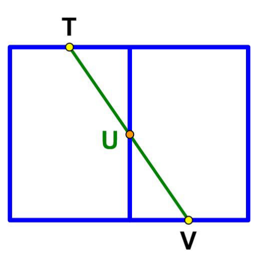

Figure 2

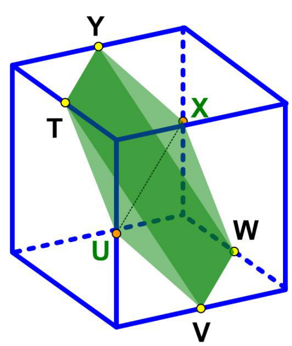

Figure 3

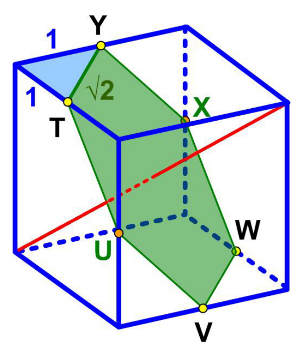

Figure 4

Now the regular hexagon is made up of six equilateral triangles (Figure 5) with sides $\sqrt{2}$. Therefore the altitude of each triangle is $\sqrt{3}/\sqrt{2}$, which means the area of each triangle is

$$\frac{1}{2}\sqrt{2}\,\frac{\sqrt{3}}{\sqrt{2}} = \frac{\sqrt{3}}{2}$$

so that the area of the hexagon is $6(\sqrt{3}/2) = 3\sqrt{3}$ or answer B.

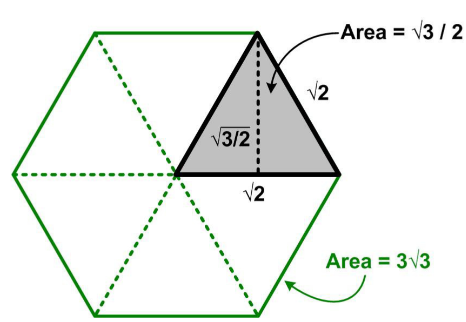

Figure 5

UKMT Solution

The UKMT solution is essentially the same as mine only with a few variations, in particular, a more detailed proof that the hexagon is regular.

The plane which goes through the points Y, T, V and W also meets the edges of the cube at the points U and X, as shown. We leave it as an exercise to check that the points U and X are also the midpoints of the edges on which they lie. The relevant cross-section is therefore the hexagon TUVWXY. We first show that this is a regular hexagon.

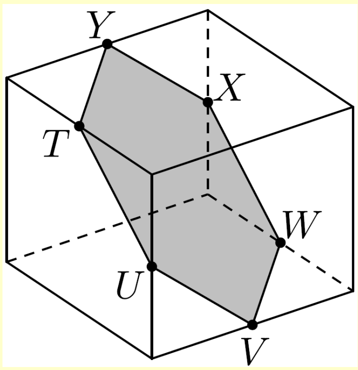

Figure 6

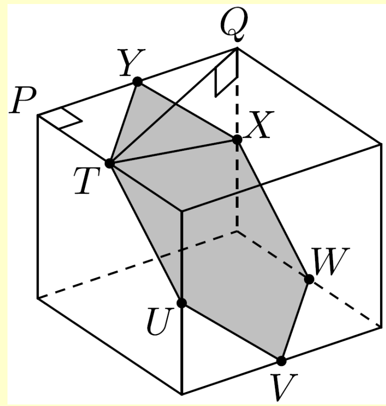

Figure 7

Let P and Q be the vertices of the cube, as shown. Applying Pythagoras' Theorem to the right-angled triangle TPY, gives

$$TY^2 = PT^2 + PY^2 = 1^2 + 1^2 = 2.$$

Therefore TY = √2. In a similar way it follows that each edge of the hexagon TUVWXY has length √2. Applying Pythagoras' Theorem to the right-angled triangle TPQ gives

$$QT^2 = PT^2 + PQ^2 = 1^2 + 2^2 = 5.$$

The triangle TXQ has a right angle at Q because the top face of the cube is perpendicular to the edge through Q and X. Hence, by Pythagoras' Theorem,

$$TX^2 = QT^2 + QX^2 = 5 + 1^2 = 6.$$

Hence TX = √6.

It follows that in the triangle TXY we have TY = Y X = √2 and TX = √6. We leave it as an exercise to check that it follows that ∠TYX = 120°. It follows similarly that each angle of TUVWXY is 120°. Hence this hexagon is regular.

The regular hexagon TUVWXY may be divided into six congruent equilateral triangles each with side length √2. We leave it as an exercise to check that the area of an equilateral triangle with side length s is $s^2\sqrt{3}/4$. It follows that equilateral triangles with side lengths √2 have area $(\sqrt{2})^2\sqrt{3}/4 = \sqrt{3}/2$.

Therefore the area of the cross-section is $6 \times \sqrt{3}/2 = 3\sqrt{3}$.

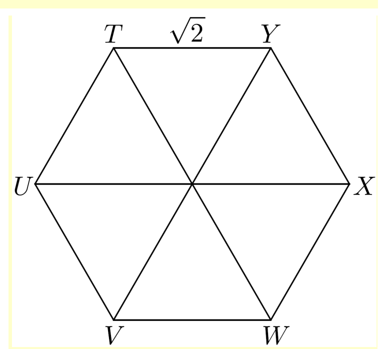

Figure 8