# Root Difference

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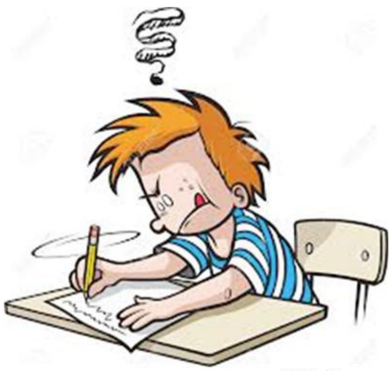

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This is another problem from the 2020 Math Calendar ([1] September).

Find the difference between the highest and lowest roots of

$$f(x) = x^3 - 54x^2 + 969x - 5780$$

## Solution

We *could* try to solve this, but an alternative would be to look at the coefficients and their relationship to the roots. Recall they are elementary symmetric functions of the roots, that is, if

$$f(x) = \mathrm{a}x^3 + \mathrm{b}x^2 + \mathrm{c}x + \mathrm{d} = 0$$

with a = 1 and roots $\lambda_1$, $\lambda_2$, $\lambda_3$, then

$$\mathrm{b} = -(\lambda_1 + \lambda_2 + \lambda_3)$$

$$\mathrm{c} = \lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_3 \lambda_1$$

$$\mathrm{d} = -(\lambda_1 \lambda_2 \lambda_3)$$

You can check this by expanding

$$f(x) = (x - \lambda_1)(x - \lambda_2)(x - \lambda_3)$$

In our problem it is clear that none of the roots are negative, since $x < 0 \Rightarrow f(x) < 0$. Given that the problem implicitly implies an order to the roots with “highest” and “lowest”, the roots must all be real.

Let us first see if we can find integral roots before worrying about rationals. So we are looking for 3 positive integers whose product is 5780 and sum is 54, to take the simplest relations first. Now

$$5780 = 4 \cdot 5 \cdot 289 = 4 \cdot 5 \cdot 17^2$$

and

$$20 + 17 + 17 = 54$$

So the difference between the highest and the lowest is 20 – 17 = 3.

## References


[1] Rapoport, Rebecca and Dean Chung, *Mathematics 2020: Your Daily epsilon of Math*, Point Rock, Quarto Publishing Group, New York, 2020