The Keyhole Problem

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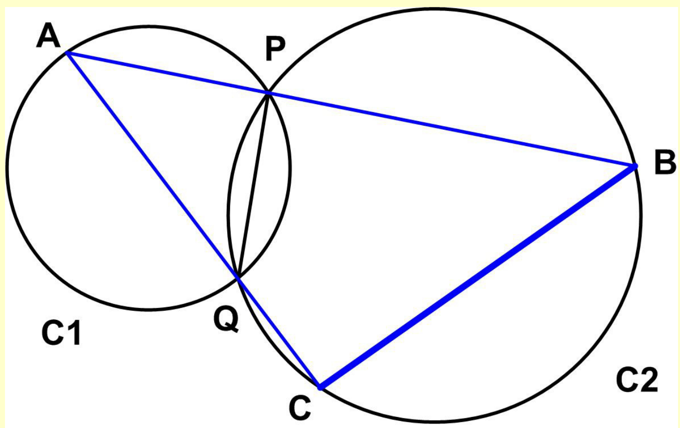
The following problem was found at the Futility Closet website:

(http://www.futilitycloset.com/2015/12/23/the-keyhole/, retrieved 2/5/16)

The Keyhole (23 December 2015)



Draw circles C1 and C2 with the common chord PQ. Now choose a point A on the arc of C1 that's outside of C2 and project it through P to B and through Q to C.

Surprisingly, the length of BC remains the same no matter where A is chosen on its arc of C1.

Solution (Case 1)

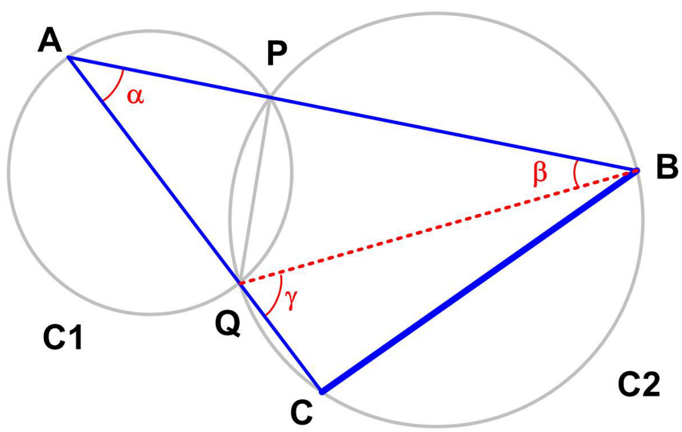


Figure 1 Solution (Case 1)

As shown in Figure 1, add the line from Q to B. Let $\alpha = \angle PAQ$, $\beta = \angle PBQ$, and $\gamma = \angle BQC$. Since angle α subtends the same arc of the small circle no matter where A is positioned, its value is always the same. Similarly for angle β at B on the larger circle. Therefore, $\gamma = \alpha + \beta$ is also constant for all positions of A. And so the arc BC, and therefore the chord BC, are also the same length for all positions of A.

Solution (Case 2) **(Update 7/1/2020)**

An email from Deb Jyoti Mitra got me to revisit this problem. In doing so, I realized the phrase "no matter where A is chosen on its arc of C1" might cause the intersection C of the line through A and Q to fall between A and Q and not outside. I wondered if the property was till true. It is!

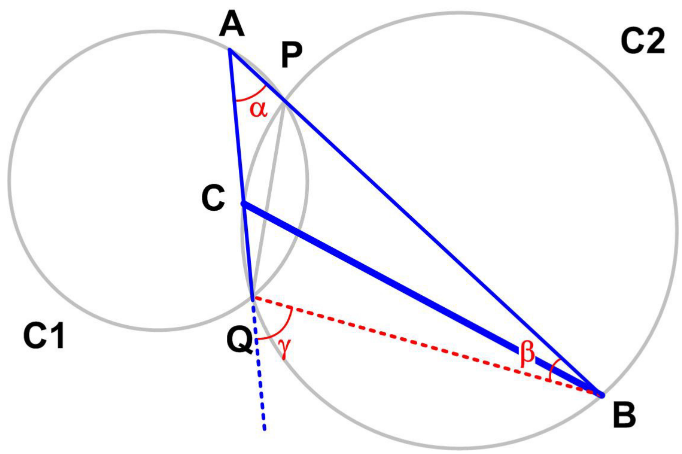


Figure 2 Solution (Case 2)

With the same parameterization as in Figure 1 we see from Figure 2 that still $\gamma = \alpha + \beta$ with α and β constant, so γ must continue to be constant, and therefore so must $180° - \gamma$, which subtends BC. Thus the chord BC is also constant in length in this case as well.

Comment 1. So the limiting case is when A = P. Then the line AB is tangent to the smaller circle C1 at P and AQ = PQ. Furthermore, C = P as well and we have the result shown in Figure 3.

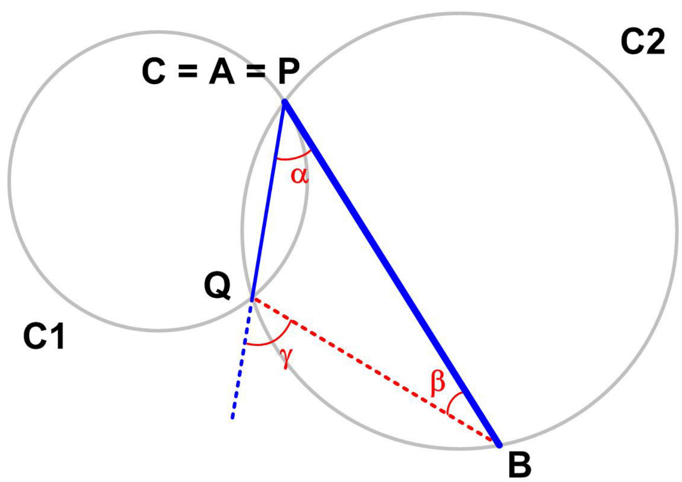


Figure 3 Solution (Limiting Case)

Comment 2. The problem is symmetric about the line through the centers of the two circles. So if A fell below the center line, we could flip the figure about the center line so that A would fall above the center line as shown in the two solutions above.

Comment 3. Mr. Mitra also pointed me to one of David Wells's books as another source for the problem in the section "angle in the same segment" ([1] p.2). Wells does not provide any proofs, nor does he appear to address the possibility that C might lie on the segment AQ.

References

[1] Wells, , *The Penguin Dictionary of Curious and Interesting Geometry*, Illustrated by John Sharp, Penguin Books, 1991