Envelope Puzzle

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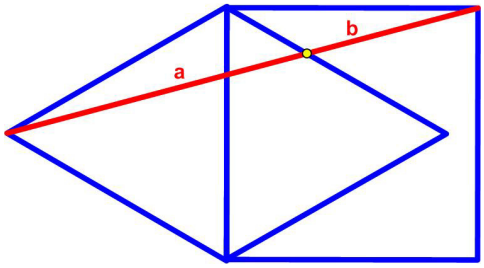

This turned out to be a challenging geometric problem from Poo-Sung Park[1] posted at the Twitter site #GeometryProblem[2]

Geometry Problem 92[3]

What is the ratio of a:b?

My Solution

Since we are only interested in the ratio a:b, we can scale the diagram by having a unit square and thus equilateral triangles with unit edges (Figure 1). It *looks* like the red line bisects the 30° angle in the left-hand equilateral triangle, but we need to prove that.

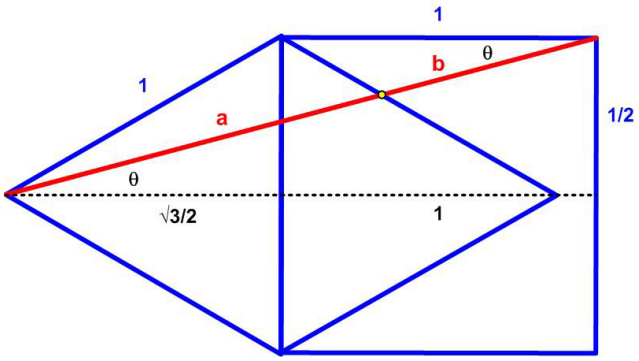

Figure 1

Now

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}.$$

So if $2\theta = 30°$, what is $x = \tan 15°$?

$$\frac{1}{\sqrt{3}} = \frac{2x}{1-x^2} \;\Rightarrow\; x^2 + 2\sqrt{3}x - 1 = 0$$

Therefore, from the quadratic formula $x = 2-\sqrt{3}$. From Figure 1 we see that $\tan\theta = (1/2)/(1 + \sqrt{3}/2) = 1/(2 + \sqrt{3}) = 2 - \sqrt{3}$. So indeed $\theta = 15°$.

Now rotate the right-hand triangle around its top vertex as shown in Figure 2. Then it follows that

$$b = \frac{1}{\sqrt{2}}.$$

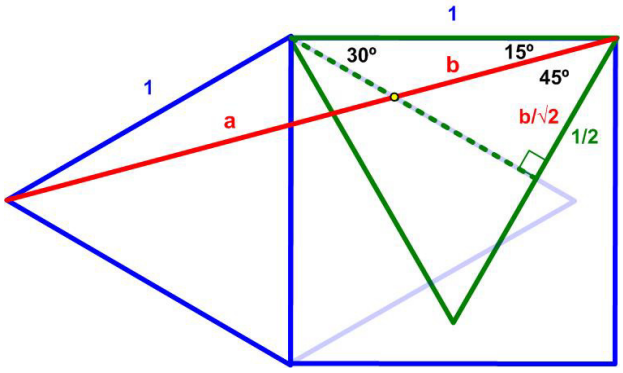

Figure 2

Going back to Figure 1 we have

$$(a+b)^2 = \frac{1}{4} + \left(\frac{\sqrt{3}}{2}+1\right)^2 = 2+\sqrt{3}$$

If we factor out b on the left-hand side and substitute $b = 1/\sqrt{2}$, then we have

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[1] https://twitter.com/puzzlist
[2] https://twitter.com/hashtag/GeometryProblem?src=hash
[3] https://twitter.com/puzzlist/status/1122342583269609474

$$(a/b + 1)^2 = 2(2 + \sqrt{3})$$

or

$$\frac{a}{b} = \sqrt{2(2+\sqrt{3})} - 1$$

Park Twitter Site Solution

When I looked at Poo-Sung Park's twitter site,[4] I found a truly clever solution by Eylem Gercek Boss.[5] He arrived at the depressingly simple result $a/b = \sqrt{3}$ as shown in Figure 3. He scaled the square to be $\sqrt{3}$ on a side and rotated an image of the left-hand triangle around its top vertex as shown. The filled-in green triangles prove to be similar, so

$$a : b = \sqrt{3} : 1$$

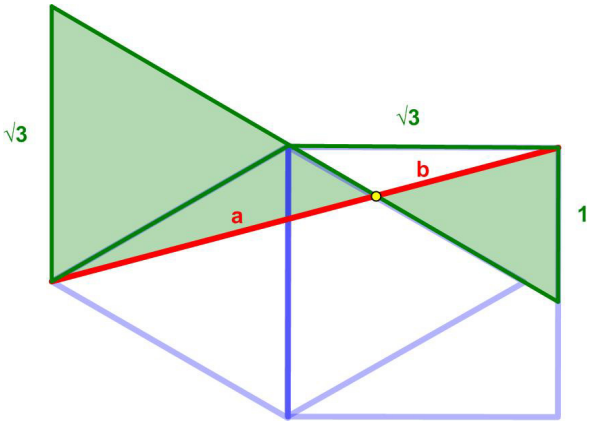

Figure 3

Comment

Can it be that my solution is also $\sqrt{3}$? Ignoring negative values,

$$\sqrt{3} = \sqrt{2(2+\sqrt{3})} - 1 \Leftrightarrow 3 = \left(\sqrt{2(2+\sqrt{3})} - 1\right)^2 = 5 + 2\sqrt{3} - 2\sqrt{2(2+\sqrt{3})}$$

$$\Leftrightarrow 2(1+\sqrt{3}) = 2\sqrt{4+2\sqrt{3}}$$

$$\Leftrightarrow (1+\sqrt{3})^2 = 4 + 2\sqrt{3}$$

$$\Leftrightarrow 4 + 2\sqrt{3} = 4 + 2\sqrt{3} \quad \checkmark$$

The crazy radicals strike again.

[4] https://twitter.com/puzzlist/status/1122342583269609474
[5] https://twitter.com/_eylem_99/status/1122351398971224065