Three Counting Puzzles

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Mirror, Mirror

Which of these two columns is biggest of them all?

987654321	123456789
087654321	123456780
007654321	123456700
000654321	123456000
000054321	123450000
000004321	123400000
000000321	123000000
000000021	120000000
+000000001	+100000000

Nous Like Gauss

The 24 four-digit numbers that include 1, 2, 3, and 4 are as follows.

1234	2314	3412
1243	2341	3421
1324	2413	4123
1342	2431	4132
1423	3124	4213
1432	3142	4231
2134	3214	4312
2143	3241	4321

Add them all up please.

That's Sum Table

And now, in two dimensions. What's the sum?

1	2	3	4	5	6	7	8	9	10
2	3	4	5	6	7	8	9	10	11
3	4	5	6	7	8	9	10	11	12
4	5	6	7	8	9	10	11	12	13
5	6	7	8	9	10	11	12	13	14
6	7	8	9	10	11	12	13	14	15
7	8	9	10	11	12	13	14	15	16
8	9	10	11	12	13	14	15	16	17
9	10	11	12	13	14	15	16	17	18
10	11	12	13	14	15	16	17	18	19

Here are three counting puzzles from Alex Bellos's book, *Can You Solve My Problems?* ([1]). Bellos recalls the famous legend of the young Gauss in the 19th century who summed up the whole numbers from 1 to 100 by finding a pattern that would simplify the work. Bellos also mentioned that Alcuin some thousand years earlier had discovered a similar, but different, pattern to sum up the numbers. In presenting these three problems he said, "The lesson … is this: If you're asked to add up a whole bunch of numbers, don't undertake the challenge literally. Look for the pattern and use it to your advantage."

Solutions

Mirror, Mirror. The sums are identical. Written out in powers of 10, the sum of the left hand column is

$$(1\cdot 9)10^8 + (2\cdot 8)10^7 + (3\cdot 7)10^6 + (4\cdot 6)10^5 + (5\cdot 5)10^4 + (6\cdot 4)10^3 + (7\cdot 3)10^2 + (8\cdot 2)10^1 + (9\cdot 1)10^0$$

and the sum of the right column is

$$(9\cdot 1)10^8 + (8\cdot 2)10^7 + (7\cdot 3)10^6 + (6\cdot 4)10^5 + (5\cdot 5)10^4 + (4\cdot 6)10^3 + (3\cdot 7)10^2 + (2\cdot 8)10^1 + (1\cdot 9)10^0$$

which are identical results, since multiplication of whole numbers commutes, that is, $m\cdot n = n\cdot m$ for any whole numbers m and n.

Nous Like Gauss. Again, let's write the sum of the 24 numbers in powers of 10 as follows:

$$(a_1 + a_2 + \ldots + a_{24})10^3 + (b_1 + b_2 + \ldots + b_{24})10^2 + (c_1 + c_2 + \ldots + c_{24})10^1 + (d_1 + d_2 + \ldots + d_{24})10^0$$

where the as represent the first digits (coefficients of 10^3), the bs the second, and so on. But all the digits are taken from the set 1, 2, 3, 4. Furthermore, as a particular digit (coefficient of a power of 10) in the number is chosen, it remains fixed while the remaining three digits cycle through the $3\cdot 2\cdot 1 = 6$ choices of the other digits from the set 1, 2, 3, 4. Hence, after permutations, all four coefficients of powers of 10 involve the same sum of 24 numbers, namely, $6\cdot 1 + 6\cdot 2 + 6\cdot 3 + 6\cdot 4 = 6\cdot(1 + 2 + 3 + 4) = 6\cdot 10$. Thus the sum is

$$(6\cdot 10)10^3 + (6\cdot 10)10^2 + (6\cdot 10)10^1 + (6\cdot 10)10^0 = 6\cdot 10^4 + 6\cdot 10^3 + 6\cdot 10^2 + 6\cdot 10^1 = 66660$$

That's Sum Table (My Solution). I summed up the numbers diagonally as shown in Figure 1, and noticed a nice cancellation:

$$1 \cdot 1 + 2 \cdot 2 + 3^2 + \ldots + 8^2 + 9^2 + 10^2 + (10^2 - 1^2) + (10^2 - 2^2) + (10^2 - 3^2) + \ldots + (10^2 - 9^2)$$

$$= 10^2 + 9 \cdot 10^2 = 10 \cdot 10^2 = 1000$$

1·1 2·2 3·3

9·11 = (10 – 1)·(10 + 1) = $10^2 - 1^2$

8·12 = (10 – 2)·(10 + 2) = $10^2 - 2^2$

1	2	3	4	5	6	7	8	9	10
2	3	4	5	6	7	8	9	10	11
3	4	5	6	7	8	9	10	11	12
4	5	6	7	8	9	10	11	12	13
5	6	7	8	9	10	11	12	13	14
6	7	8	9	10	11	12	13	14	15
7	8	9	10	11	12	13	14	15	16
8	9	10	11	12	13	14	15	16	17
9	10	11	12	13	14	15	16	17	18
10	11	12	13	14	15	16	17	18	19

Figure 1 My Solution

That's Sum Table (Bellos Solution). You may have solved this in one of two ways. I'll call the first the Alcuin method—since it is most faithful to how he paired the numbers when summing from 1 to 100—and the second the Gauss method.

Alcuin method. Pair the numbers diagonally from top left to bottom right. You'll see that (1 + 19) = 20, (2 + 18) = 20, (3 + 17) = 20, and so on until (9 + 11) = 20. There is one of the first pair, two of the second pair, three of the third pair, and so on. So the sum of the pairs is

$$20 + (2 \times 20) + (3 \times 20) + \ldots + (9 \times 20),$$

or $$(1 + 2 + 3 + \ldots + 9) \times 20,$$

which is $$[(9 \times 10)/2 \times 20] = 45 \times 20 = 900.$$

To this we add the ten 10s in the diagonal that we haven't yet counted. So the total is 900 + 100 = 1,000.

Gauss method. The sum of the first row is equal to

$$(1 + 10) + (2 + 9) + \ldots + (5 + 6) = 5 \times 11 = 55.$$

The numbers in the second row are all +1 of the numbers in the first row, so the sum of the second row is equal to the sum of the first row plus 10. The sum of the third row is the sum of the second row plus 10, which is the sum of the first row plus 20. The sum of the whole table is therefore:

$$55 + (55 + 10) + (55 + 20) + \ldots + (55 + 90)$$

which is $$10 \times 55 + (10 + 20 + 30 + \ldots + 90)$$

or $$550 + 10(1 + 2 + 3 + \ldots + 9) = 550 + (10 \times 45) = 550 + 450 = 1000$$

References

[1] Bellos, Alex, *Can You Solve My Problems? Ingenious, Perplexing, and Totally Satisfying Math and Logic Puzzles*, Guardian Books, Faber and Faber Ltd, 2016, pp.165-169