Amazing Radical Sum

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The craziness of manipulating radicals strikes again. This 2006 four-star problem from Colin Hughes at *Maths Challenge*[1] is really astonishing, though it takes the right key to unlock it.

Problem

Consider the following sequence:

$$S(n)=\frac{1}{\sqrt{1}+\sqrt{2}}+\frac{1}{\sqrt{2}+\sqrt{3}}+\frac{1}{\sqrt{3}+\sqrt{4}}+\ldots+\frac{1}{\sqrt{n+1}+\sqrt{n}}$$

For which values of [positive integer] n is S(n) rational?

Solution

The trick is to "rationalize" the terms, that is, multiply the top and bottom by the conjugate $\sqrt{(n+1)}-\sqrt{n}$.

$$\frac{1}{\sqrt{n+1}+\sqrt{n}}\frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n+1}-\sqrt{n}}=\sqrt{n+1}-\sqrt{n}$$

Then we have another telescoping sum:

$$S(n)=\left(\sqrt{2}-\sqrt{1}\right)+\left(\sqrt{3}-\sqrt{2}\right)+\left(\sqrt{4}-\sqrt{3}\right)+\ldots+\left(\sqrt{n+1}-\sqrt{n}\right)=\sqrt{n+1}-1 \tag{1}$$

So $S(n)$ is rational if and only if

$$\sqrt{n+1}-1=\frac{k}{m} \tag{2}$$

for some positive integers k and m, $k > m$.

Claim. k/m is a positive integer.[2]

Suppose not, that is, suppose $m > 1$ and m does not divide k. Then $k = mq + r$ for $0 < r < m$ and some integer q. From equation (2)

$$n=\left(\frac{k}{m}+1\right)^2-1=\frac{k}{m}\left(\frac{k}{m}+2\right)=\frac{mq+r}{m}\left(\frac{mq+r}{m}+2\right)=q(q+2)+\frac{r}{m}\left(2(q+1)+\frac{r}{m}\right) \tag{3}$$

Assume that r/m is reduced to lowest terms, that is, r and m have no common factors. Then for equation (3) to represent an integer n, all of m must divide the last term $2(q + 1) + r/m$, which must therefore at least be an integer itself, say p. But that would mean $r/m = p - 2(q + 1)$ is also an integer, contradicting the fact that $r/m < 1$. Hence k/m must be a positive integer. Q.E.D.

Therefore $S(n)$ is integral, if and only if n is one less than a perfect square, that is, $n = k^2 - 1$ for some integer k (or equivalently, if n is the product of two integers that differ by 2 (consecutive odd

[1] "Reciprocal Radical Sum" Problem ID: 267 (05 Feb 2006) Difficulty: 4 Star at mathschallenge.net "A four-star problem: A comprehensive knowledge of school mathematics and advanced mathematical tools will be required." (https://mathschallenge.net/problems/pdfs/mathschallenge_4_star.pdf)

[2] JOS: *Maths Challenge* does not show this.

numbers or consecutive even numbers), since $k^2 - 1 = (k + 1)(k - 1)$). In which case, from equation (1), $S(n) = \sqrt{n+1} - 1 = \sqrt{(k^2 - 1) + 1} - 1 = k - 1$.

For example, if $n = 3 = 3 \cdot 1$, then

$$S(3) = \frac{1}{\sqrt{1} + \sqrt{2}} + \frac{1}{\sqrt{2} + \sqrt{3}} + \frac{1}{\sqrt{3} + \sqrt{4}} = \sqrt{3+1} - 1 = 1.$$

And $S(8) = S(4 \cdot 2) = 2$, $S(15) = S(5 \cdot 3) = 3$, etc. Simply amazing!

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