Lopsided Hexagon Problem

26 November 2019

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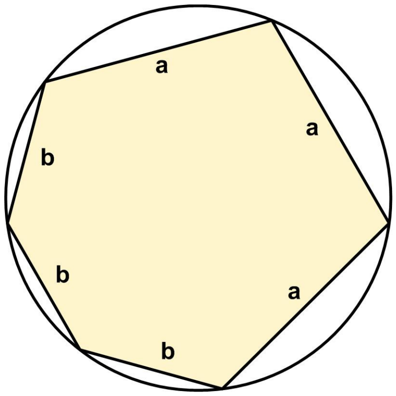


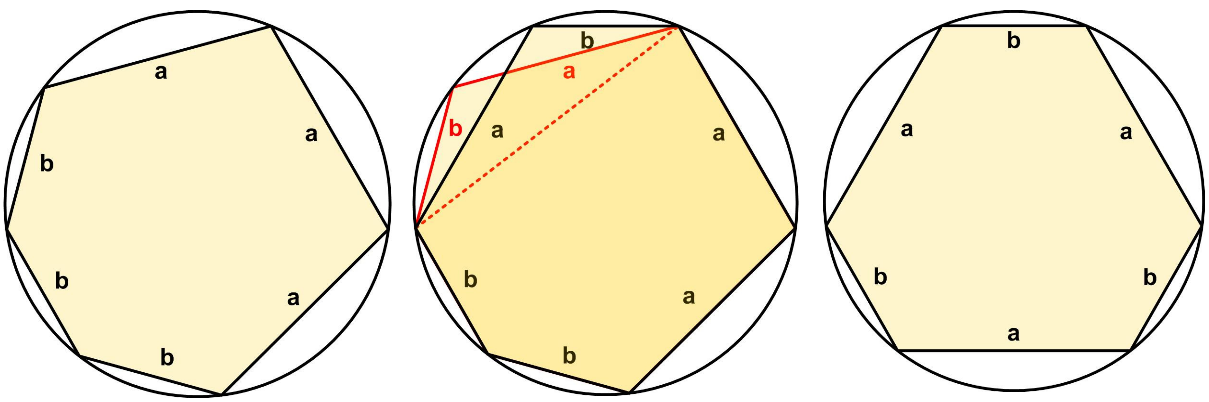
Here is another good problem from *Five Hundred Mathematical Challenges* ([1]):

Problem 100. A hexagon inscribed in a circle has three consecutive sides of length a and three consecutive sides of length b. Determine the radius of the circle.

This problem made me think of the Putnam Octagon Problem (http://josmfs.net/2019/04/04/putnam-octagon-problem/). Again my approach might be considered a bit pedestrian. 500 Math Challenges had a slightly slicker solution.

My Solution

As with the Putnam Octagon Problem, we swap adjacent a-b sides (Figure 1(a)-(b)) with the new vertex still on the circle. Doing this twice produces an inscribed hexagon (Figure 1(c)) with



(a) (b) (c)

Figure 1 Transforming Original Hexagon to One with Alternating Sides

alternating a-b sides. What this accomplishes is giving us a hexagon with equal angles θ at each vertex (Figure 2).

Using techniques from before (such as, sliding an arrow around the perimeter, making 6 turns of $180 - \theta$ degrees each that add up to 360°, which means $180° - \theta = 60°$), we have $\theta = 120°$.

Now consider the additions to the hexagon shown in Figure 3. We have dropped perpendiculars from the center of the circle to a side of length a and to an adjacent side of length b. The radii of the circle to the vertices of the hexagon mean the resulting triangles on bases a and b are isosceles. This means their base angles are equal and the two sub-right triangles are congruent. Therefore the altitudes are also perpendicular bisectors of their

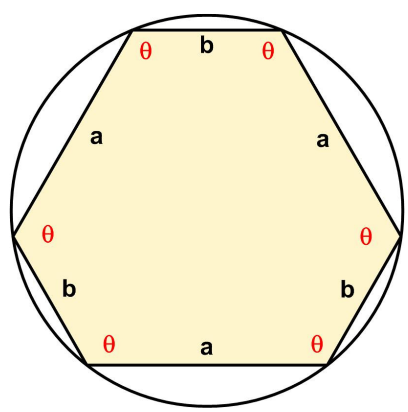


Figure 2 Equal Vertex Angles

bases.

The base angle of the triangle with base a is α, and the base angle of the triangle with base b is β. This means

$$\alpha + \beta = \theta = 120°.$$

and so we use the cosine of the sum of angles identity

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$$

Now

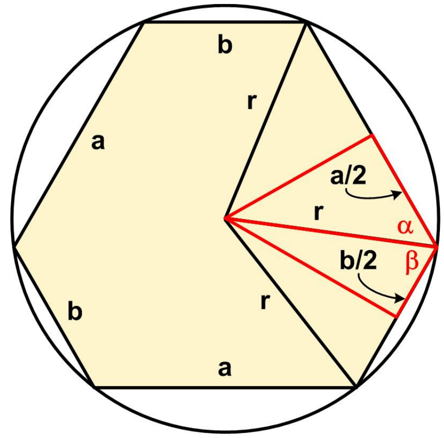


Figure 3 My Solution

$$\cos(\alpha+\beta) = \cos 120° = -\frac{1}{2}$$

$$\cos\alpha = \frac{a}{2r}, \quad \cos\beta = \frac{b}{2r}$$

$$\sin\alpha = \sqrt{1-\left(\frac{a}{2r}\right)^2} = \frac{1}{2r}\sqrt{4r^2-a^2}, \quad \sin\beta = \sqrt{1-\left(\frac{b}{2r}\right)^2} = \frac{1}{2r}\sqrt{4r^2-b^2}$$

Therefore, from the cosine of the sum of angles identity,

$$-\frac{1}{2} = \frac{ab}{4r^2} - \frac{1}{4r^2}\sqrt{4r^2-a^2}\sqrt{4r^2-b^2}$$

$$ab + 2r^2 = \sqrt{4r^2-a^2}\sqrt{4r^2-b^2}$$

$$\left(ab+2r^2\right)^2 = \left(4r^2-a^2\right)\left(4r^2-b^2\right)$$

$$12r^4 = 4r^2(a^2+ab+b^2)$$

$$\boxed{r = \sqrt{\frac{a^2+ab+b^2}{3}}}$$

As a check, if $a = b$, then $r = a$, which agrees with the result when the hexagon is regular (equilateral).

500 Math Challenges Solution

One can rearrange the sides of the hexagon so each pair of consecutive sides are of length a and b. Since all the angles of $ABCDEF$ are congruent, each is 120°.[1]

By the law of cosines (Figure 4)

$$\overline{AC}^2 = 2r^2\left(1-\cos 120°\right) = a^2 + b^2 - 2ab\cos 120°$$

or[2]

$$3r^2 = a^2 + b^2 + ab$$

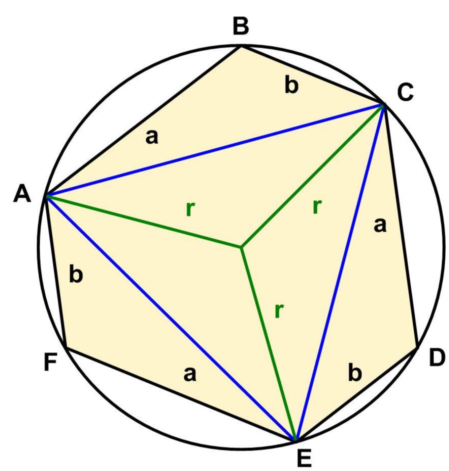


Figure 4 500 Math Challenges Solution

[1] JOS: A few steps were omitted here!

[2] JOS: Since the blue triangle is equilateral and the green triangles have equal sides, they are all congruent. Therefore their vertex angles are equal and add up to 360°. Thus each vertex angle is 120°.

500 Math Challenges use of the law of cosines definitely streamlined the computations. Again I missed the simpler approach.

References

[1] Barbeau, Edward J., Murray S. Klamkin, William O. J. Moser, *Five Hundred Mathematical Challenges*, Spectrum Series, Mathematical Association of America, Washington D.C, 1995